Monopoles in arbitrary dimension

3rd International Young Researchers Workshop on Mechanics, Geometry, and Control

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Plan of the talk

1. Introduction.
2. Homogeneous principal bundles and invariant connections. Wang maps and canonical connections.
3. Chern-Weil homomorphism. The charge of a monopole.
4. Examples: $SO(2n)$-monopoles and the Yang monopole.

Details available at

Díaz, P. and Lázaro-Camí, J.-A. Monopoles in arbitrary dimension. CRM preprint **838**. `math hep-th/0811.4187`

Introduction

- Monopoles are a certain type of principal connections on principal bundles over $\mathbb{R}^m$ satisfying Yang-Mills equations and some additional requirements:
  1. They are static spherically symmetric: time-independent and $SO(m)$-invariant.
  2. One may associate them a non-zero charge through their field strength. Monopoles will exist only in odd dimensional spaces, $m = 2n + 1$.

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- Purpose: build principal bundles over $\mathbb{S}^{m-1} = SO(m)/SO(m-1)$ admitting a $SO(m)$ (left) action.

Yang-Mills theories on principal bundles

Let $\pi : P \to M$ be a principal bundle with structural group G. Let $p \in P$, $g \in G$, and $\eta \in \mathfrak{g}$.

- A **principal connection** or **gauge potential** $\omega \in \Omega^1(P; \mathfrak{g})$ is a one form on P with values in the Lie algebra $\mathfrak{g}$ of G such that

$$R_g^* \omega = \mathrm{Ad}_{g^{-1}} \omega,$$
$$\omega(p) \left(\left. \frac{d}{dt} \right|_{t=0} R_{\exp(t\eta)}(p) \right) = \eta.$$

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- The **horizontal space** at $p \in P$, $\mathrm{Hor}_p = \ker \omega$, is such that $T_p P = \mathrm{Hor}_p \oplus \mathrm{Ver}_p$, where $\mathrm{Ver}_p = \ker T_p \pi$.

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- Let M be a Riemann manifold. ω is a **Yang-Mills connection** if

$$\delta_\omega \Omega^\omega = -(-1)^{\dim(M)} * \circ D^\omega \circ * \Omega^\omega = 0.$$

Homogeneous principal bundles

Definition

Let K and G be two Lie groups and $H \subset K$ a closed subgroup. A **homogeneous principal bundle** $\pi : P \to K/H$ with structural group G is a principal bundle over a homogeneous space K/H together with a left K-action on P by automorphisms which projects to the left multiplication of K on the base manifold K/H.

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- Homogeneous principal bundles $\pi : P \to K/H$ are (modulo isomorphisms) in one-to-one correspondence with group homomorphisms $\lambda : H \to G$ (modulo conjugation) so that $\pi : P \to K/H$ is isomorphic to

$$P_\lambda := K \times_H G = \left\{ \left(kh, \lambda(h)^{-1} g \right) \in K \times G \mid h \in H \right\}.$$

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$$h \cdot p = p \cdot \lambda(h), \quad p \in \pi^{-1}([e]), \quad h \in H.$$

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- P_λ is isomorphic to the homogeneous space $(K \times G)/\widetilde{H}$, where $\widetilde{H}$ is the closed subgroup $\widetilde{H} = \{(h, \lambda(h)) \mid h \in H\} \subset K \times G$, clearly isomorphic to H.

Invariant connections on homogeneous principal bundles

Definition

Let $\pi : P_\lambda \to K/H$ be a homogeneous principal bundle with left action $L_\lambda : K \times P_\lambda \to P_\lambda$. We say that a principal connection ω is K-invariant if $(L_\lambda)_k^* \omega = \omega$ for any $k \in K$.

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Proposition

*Let $\mathfrak{k}$, $\mathfrak{h}$, and $\mathfrak{g}$ denote the Lie algebra of K, H, and the gauge group G respectively. K-invariant principal connections on $\pi : P_\lambda \to K/H$ are in one-to-one correspondence with linear maps (**Wang maps**) $W : \mathfrak{k} \to \mathfrak{g}$ such that*

1. $W(\xi) = T_e\lambda(\xi)$ *for any* $\xi \in \mathfrak{h}$,
2. $W(\mathrm{Ad}_h \xi) = \mathrm{Ad}_{\lambda(h)}(W(\xi))$ *for any* $\xi \in \mathfrak{k}$ *and any* $h \in H$.

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Definition

A homogeneous space K/H is called **reductive** if $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$ and $\mathrm{Ad}_h(\mathfrak{m}) \subseteq \mathfrak{m}$. For a reductive homogeneous space K/H, the linear map $\mathbf{W} : \mathfrak{k} \to \mathfrak{g}$ defined as $\mathbf{W}|_{\mathfrak{h}} = T_e\lambda$ and $\mathbf{W}|_{\mathfrak{m}} = 0$ is called the **canonical connection**.

Example

It can be shown that the principal H-bundle $K \to K/H$ admits a K-invariant connection if and only if K/H is reductive. The canonical connection $\omega \in \Omega^1(K, \mathfrak{h})$ on $K \to K/H$ is given by the $\mathfrak{h}$-valued part of the **Maurer-Cartan form** ω_{MC} which is defined by $\omega_{MC}(k)(\xi_K(k)) = \xi \in \mathfrak{k}$, $k \in K$. That is, $\omega(k)(\xi_K(k)) = \mathrm{proj}_{\mathfrak{h}}(\xi)$.

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- Principal connections can be understood as a one form $\Phi \in \Omega^1(P; VP)$, $VP = \cup_{p \in P} \mathrm{Ver}_p$,

$$\Phi_p(X) = T_e R_p \circ \omega_p(X), \quad X \in \mathfrak{X}(P).$$

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- Using $TP_\lambda = TK \times_{TH} TG$, the connection Φ_λ induced from Φ is defined by the following commutative diagram:

$$\begin{array}{ccc} TK \times TG & \xrightarrow{\Phi \times \mathrm{Id}} & TK \times TG \\ {\scriptstyle Tq} \downarrow & & \downarrow {\scriptstyle Tq} \\ TK \times_{TH} TG & \xrightarrow[\Phi_\lambda]{} & TK \times_{TH} TG = T(K \times_H G), \end{array}$$

where $q : K \times G \to K \times_H G$ sends each element to its corresponding equivalent class in $K \times_H G$ and Tq is its tangent map.

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Let K/H be reductive. Then, the canonical connection on P_λ is induced from the canonical connection of $K \longrightarrow K/H$.

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Example

If K/H is reductive, then $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$ and $\mathrm{Ad}_h(\mathfrak{m}) \subseteq \mathfrak{m}$ for any $h \in H$. The field strength $\Omega^{\mathbf{W}}$ associated to the Wang map (canonical connection)

$$\mathbf{W}(\xi) = \begin{cases} T_e\lambda(\xi) \text{ if } \xi \in \mathfrak{h} \\ 0 \text{ if } \xi \in \mathfrak{m}. \end{cases}$$

is given by $\Omega^{\mathbf{W}}([e])(v_1, v_2) = -T_e\lambda\left(\mathrm{proj}_{\mathfrak{h}}([v_1, v_2])\right)$, $v_1, v_2 \in \mathfrak{m}$. The n-dimensional sphere $\mathbb{S}^n$ is reductive, where $K = SO(n+1)$,

$$H = \begin{pmatrix} 1 & 0 \\ 0 & SO(n) \end{pmatrix}, \text{ and } \mathfrak{m} = \left\{ \begin{pmatrix} 0 & -v^\top \\ v & 0 \end{pmatrix} \mid v \in \mathbb{R}^n \right\}.$$

Symmetric spaces

Definition

A triple (K, H, σ) is a **symmetric space** if K, H are Lie groups, $H \subset K$, $\sigma : K \longrightarrow K$ is an involutive automorphism, and $K_\sigma^e \subseteq H \subseteq K_\sigma$. Here K_σ denotes the set of elements of K which are invariant by σ and K_σ^e the identity component of K_σ.

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- At the level of Lie algebras, $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$ and

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$$

  where $\mathfrak{h}$ is the eigenspace of $T_e\sigma : \mathfrak{k} \longrightarrow \mathfrak{k}$ associated to the eigenvalue 1 and $\mathfrak{m}$ is the eigenspace associated to -1.

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  where $\mathfrak{h}$ is the eigenspace of $T_e\sigma : \mathfrak{k} \longrightarrow \mathfrak{k}$ associated to the eigenvalue 1 and $\mathfrak{m}$ is the eigenspace associated to -1.
- When $G \subseteq GL(n; \mathbb{R})$, $P_\lambda \longrightarrow K/H$ can be regarded as subbundles of the frame bundle. Then, any K-invariant principal connection on P_λ (i.e., a Wang map $W : \mathfrak{k} \longrightarrow \mathfrak{g} \subset \mathfrak{gl}(n; \mathbb{R})$) can be consequently considered as K-invariant affine connection (i.e., a Wang map $W : \mathfrak{k} \longrightarrow \mathfrak{gl}(n; \mathbb{R})$).

Invariant connections on symmetric spaces

Theorem

Let K be a simple Lie group and (K, H, σ) a symmetric space. The set of K-invariant affine connections on K/H consists of just the canonical connection in all cases except for the following:

$$\begin{array}{rl} SU(n)/SO(n) & n \geq 3, \\ SU(2n)/SP(n) & n \geq 3, \\ E_6/F_4. & \end{array}$$

Each of these spaces has a one-dimensional family of invariant affine connections.

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The canonical connection on a symmetric space is a Yang-Mills connection.

Chern-Weil homomorphism

- Let $S(\mathfrak{g}^*)^G = \oplus_{k\geq 0} S^k(\mathfrak{g}^*)^G$ be the symmetric algebra of multilinear functions on $\mathfrak{g}$ which are Ad-invariant. That is, $f \in S^k(\mathfrak{g}^*)^G$ if $f : \mathfrak{g} \times \overset{k)}{\dots} \times \mathfrak{g} \to \mathbb{R}$ (or $\mathbb{C}$) and

$$\begin{aligned} f(\eta_{\sigma(1)}, ..., \eta_{\sigma(k)}) &= f(\eta_1, ..., \eta_k), \\ f(\mathrm{Ad}_g(\eta_1), ..., \mathrm{Ad}_g(\eta_k)) &= f(\eta_1, ..., \eta_k) \end{aligned}$$

for any permutation $\sigma \in S_k$, any $g \in G$, and any $\eta_1, ..., \eta_k \in \mathfrak{g}$.

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- $S(\mathfrak{g}^*)^G \cong P(\mathfrak{g}^*)^G = \oplus_{k\geq 0} P^k(\mathfrak{g}^*)^G$, the algebra of Ad-invariant homogeneous polynomials on $\mathfrak{g}$, through *the polarization formula.*

$$\mathrm{Sym}(f)(\eta_1, ..., \eta_k) = \frac{1}{k!} \sum_{i=0}^{k-1} (-1)^i \sum_{j_r \neq j_s} f\left(\eta_{j_1} + ... + \eta_{j_{k-i}}\right).$$

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- Let $\omega \in \Omega^1(P; \mathfrak{g})$ be a principal connection on $\pi : P \to M$ and Ω^ω its curvature. The $2k$-form $\bar{f}(\Omega^\omega)(p)(X_1, ..., X_{2k}) =$
$$\frac{1}{2^k} \sum_{\sigma \in S_{2k}} (-1)^{|\sigma|} f\left(\Omega^\omega(p)(X_{\sigma(1)}, X_{\sigma(2)}), ..., \Omega^\omega(p)(X_{\sigma(2k-1)}, X_{\sigma(2k)})\right)$$
is G-invariant and horizontal.

Chern-Weil homomorphism

- There exists a uniquely defined $2k$-form $cw\left(f, P, \omega\right) \in \Omega^{2k}\left(M\right)$ such that

$$\pi^{*}\left(cw\left(f, P, \omega\right)\right) = \bar{f}\left(\Omega^{\omega}\right).$$

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$$\pi^*(cw\,(f, P, \omega)) = \bar{f}\,(\Omega^{\omega})\,.$$

- $cw\,(f, P, \omega)$ is closed, so there is a well defined *de Rham* cohomology class $[cw\,(f, P, \omega)] \in H^{2k}\,(M)$, called the **characteristic class** of f, which is independent of the particular choice of $\omega \in \Omega^1(P; \mathfrak{g})$.

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- The mapping

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- The characteristic classes of a trivial principal bundle all vanish.

Chern-Weil homomorphism

- There exists a uniquely defined $2k$-form $cw(f, P, \omega) \in \Omega^{2k}(M)$ such that

$$\pi^*(cw(f, P, \omega)) = \bar{f}(\Omega^\omega).$$

- $cw(f, P, \omega)$ is closed, so there is a well defined *de Rham* cohomology class $[cw(f, P, \omega)] \in H^{2k}(M)$, called the **characteristic class** of f, which is independent of the particular choice of $\omega \in \Omega^1(P; \mathfrak{g})$.
- The mapping

$$\begin{array}{rcl} Cw_P : S(\mathfrak{g}^*)^G & \longrightarrow & H^*(M) \\ f & \longmapsto & [cw(f, P, \omega)] \end{array}$$

  is a homomorphism of commutative algebras (**Chern-Weil homomorphism**).
- The characteristic classes of a trivial principal bundle all vanish.
- If $f \in S^n(\mathfrak{g}^*)^G$, the quantity

$$Q := \int_{\mathbb{S}^{2n}} cw(f, P_\lambda, \omega) = \mathbf{d}\,\mathrm{vol}(\mathbb{S}^{2n})$$

  is called **the charge** of the monopole.

Example: SO(2n)-monopoles

Let $2n \geq 6$, $G = SO(2n)$, $n \in \mathbb{N}$, and $o = (1, 0, \overset{2n)}{\ldots}, 0) \in \mathbb{S}^{2n}$. Let $\{\xi_{\alpha,\beta}\}_{\alpha>\beta}$, $\alpha, \beta \in \{1, ..., n\}$, be the basis of $\mathfrak{so}(n)$ such that $\xi_{\alpha,\beta}$ is the matrix whose entries are 1 in the position (α, β), -1 in the position (β, α), and 0 elsewhere.

- There are three different $\lambda : SO(2n) \to SO(2n)$: the trivial homomorphism, the identity $\mathrm{Id} : SO(2n) \to SO(2n)$, whose associated principal bundle is $SO(2n+1) \to SO(2n+1)/SO(2n)$, and the conjugation δ by the diagonal matrix $(-1, ..., -1, 1)$. Observe that $\delta \in O(2n)$ but $\delta \notin SO(2n)$.

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- The field strength $\Omega^{\mathbf{W}}_{\mathrm{Id}}$ associated to $\mathrm{Id} : SO(2n) \to SO(2n)$ is such that

$$\Omega^{\mathbf{W}}_{\mathrm{Id}}(o)\left(v_i, v_j\right) = -\xi_{ji}, \quad v_i, v_j \in \mathbb{R}^{2n}, \quad i < j.$$

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$$\Omega^{\mathbf{W}}_{\delta}(o)\left(v_i, v_j\right) = -\xi_{j,i} \text{ if } i < j,\ j \neq 2n, \text{ and } \Omega^{\mathbf{W}}_{\delta}(o)\left(v_i, v_{2n}\right) = \xi_{2n,i}.$$

Example: SO(2n)-monopoles

Definition

The Pfaffian Pf is the unique (up to a sign) polynomial function on $SO(2n)$ such that $\mathrm{Pf}^2(A) = \det(A)$, $A \in SO(2n)$. The **Euler class** $\chi(P)$ is defined as $\frac{1}{\pi^n} Cw_P\,(\mathrm{Sym}(\mathrm{Pf}))$.

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Proposition

The Euler class associates a non-zero charge to $\pi_{\mathrm{Id}} : P_{\mathrm{Id}} \to \mathbb{S}^{2n}$ and $\pi_\delta : P_\delta \to \mathbb{S}^{2n}$ (if n is odd, up to a constant factor, it is the unique characteristic class doing so). More concretely,

$$\frac{1}{\pi^n} \int_{\mathbb{S}^{2n}} cw\left(\mathrm{Sym}(\mathrm{Pf}), P_{\mathrm{Id}}, \boldsymbol{\omega}\right) = 2, \quad \frac{1}{\pi^n} \int_{\mathbb{S}^{2n}} cw\left(\mathrm{Sym}(\mathrm{Pf}), P_\delta, \boldsymbol{\omega}\right) = -2.$$

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Gibbons, G. W. and Townsend, P. K. Self-gravitating Yang monopoles in all dimensions. *Class. Quantum Grav.* **23**, pp. 4873–4885 (2006)

compute the charge of the monopole as $\int_{\mathbb{S}^{2n}} \mathrm{trace}(\Omega^\omega \wedge \overset{n)}{\ldots} \wedge \Omega^\omega)$ which is zero!

Yang monopole

According to

Yang, C. N. Generalization of Dirac's monopole to SU_2 gauge fields. *J. Math. Phys.* **19** (1), pp. 320-328 (1978),

there exist two non-trivial principal bundles $\pi_i : P \to \mathbb{S}^4 \cong SO(5)/SO(4)$, $i = 1, 2$, with gauge group $G = SU(2)$ admitting a $SO(5)$-invariant principal connection.

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Proposition

The unique homomorphism $\lambda : SO(4) \to SU(2)$ is the trivial one. There exist two non-equivalent homomorphisms $\sigma_i : Spin(4) \cong S^3 \times S^3 \to SU(2) \cong S^3$, $i = 1, 2$, given by $\sigma_1(x, y) = x$ and $\sigma_2(x, y) = y$. Therefore, two principal bundles $\pi_i : P \to \mathbb{S}^4 \cong Spin(5)/Spin(4)$, $i = 1, 2$, with gauge group $SU(2)$ admitting a (left) $Spin(5)$-action.

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