The Skinner-Rusk formalism in higher field theories

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Outline

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Symplectic Geometry

Definition

$\Omega \in \Lambda^2 M$ is symplectic iff

- nondegenerate: $i_v \Omega = 0 \Leftrightarrow v = 0$,
- closed: $\mathrm{d}\Omega = 0$.

- $m = \dim M = 2n$
- (M, Ω) is called a symplectic manifold.
- $\Omega^n / n!$ is the symplectic volume (or the Liouville measure).
- Musical isomorphisms: $TM \underset{\sharp}{\overset{\flat}{\rightleftarrows}} T^*M$.

Theorem (Darboux)

$$\Omega = \sum_{i=1}^{n} \mathrm{d}q^i \wedge \mathrm{d}p_i$$

Cosymplectic Geometry

Definition

(M^{2n+1}, Ω, μ) is cosymplectic iff

- nondegenerate: $\Omega^n \wedge \mu \neq 0$,
- closed: $\mathrm{d}\Omega = 0$ and $\mathrm{d}\mu = 0$.

The musical isomorphism $\flat : \mathfrak{X}M \longrightarrow \Lambda M$ is given by

$$\flat(v) := i_v\Omega + \mu(v)\mu.$$

The Reeb vector field $R := \flat^{-1}(\mu)$ is given by the equations:

$$i_R\Omega = 0 \quad \text{and} \quad i_R\mu = 1.$$

Theorem (Darboux)

$$\Omega = \sum_{i=1}^{n} \mathrm{d}q^i \wedge \mathrm{d}p_i \quad \textit{and} \quad \mathrm{d}t = \mu$$

(Co)symplectic examples

Example (T^*Q as a symplectic manifold)

Let $M = T^*Q$ and $\Omega = -\mathrm{d}\Theta$, where Θ is the Liouville one-form

$$\Theta = p_i \mathrm{d}q^i.$$

Then (T^*Q, Ω) is a symplectic manifold and

$$\Omega = \mathrm{d}q^i \wedge \mathrm{d}p_i.$$

Example ($\mathbb{R} \times T^*Q$ as a cosymplectic manifold)

Let $M = \mathbb{R} \times T^*Q$, $\Omega = \mathrm{d}q^i \wedge \mathrm{d}p_i$ and $\mu = \mathrm{d}t$. Then $(\mathbb{R} \times T^*Q, \Omega, \mu)$ is a cosymplectic manifold.

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Mechanics (the symplectic case)

1. The Lagrangian $L : TQ \longrightarrow \mathbb{R}$.
2. The Legendre transform $FL : TQ \longrightarrow T^*Q$,
   $FL(q^i, \dot{q}^i) = (q^i, \hat{p}_i = \frac{\partial L}{\partial \dot{q}^i})$.
3. The Poincaré-Cartan 1-form $\Theta_L := FL^*\Theta = \frac{\partial L}{\partial \dot{q}^i} \mathrm{d}q^i$.
4. The Poincaré-Cartan 2-form $\Omega_L := -\mathrm{d}\Theta_L = \mathrm{d}q^i \wedge \mathrm{d}\hat{p}_i$.
5. (TQ, Ω_L) is symplectic whenever L is regular.
6. The Lagrangian energy $E_L (= \hat{p}) := \dot{q}^i \hat{p}_i - L = \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L$.
7. A Lagrangian vector field $X \in \mathfrak{X}(TQ)$ satisfies $i_X \Omega_L = \mathrm{d}E$.

Theorem

Let $X \in \mathfrak{X}(TQ)$ be a second-order vector field. The integral curves of X satisfy the Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0.$$

Mechanics (the cosymplectic case)

1. The Lagrangian $L : \mathbb{R} \times TQ \longrightarrow \mathbb{R}$.
2. The Legendre transform $FL : \mathbb{R} \times TQ \longrightarrow \mathbb{R} \times T^*Q$,
   $FL(t, q^i, \dot{q}^i) = (t, q^i, \hat{p}_i = \frac{\partial L}{\partial \dot{q}^i})$.
3. The Hamiltonian energy $H = (FL^{-1})^* E_L = \dot{q}^i p_i - L \circ FL^{-1}$.
4. The forms $\Theta_H := \Theta - H\mathrm{d}t$ and $\Omega_H := \Omega + \mathrm{d}H \wedge \mathrm{d}t$.
5. The Poincaré-Cartan 1-form $\Theta_L := FL^*\Theta_H = \frac{\partial L}{\partial \dot{q}^i}\mathrm{d}q^i - E_L\mathrm{d}t$.
6. The Poincaré-Cartan 2-form $\Omega_L := -\mathrm{d}\Theta_L = \mathrm{d}q^i \wedge \mathrm{d}\hat{p}_i - \wedge\mathrm{d}t \wedge \mathrm{d}\hat{p}$.
7. $(\mathbb{R} \times TQ, \Omega_L, \mathrm{d}t)$ is cosymplectic whenever L is regular.

Theorem

The integral curves of the Reeb vector field satisfy the Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0.$$

Multisymplectic geometry

Definition

$\Omega \in \Lambda^k M$ is multisymplectic iff

- nondegenerate: $i_v\Omega = 0 \Leftrightarrow v = 0$,
- closed: $\mathrm{d}\Omega = 0$.

- (M, Ω) is called a multisymplectic manifold.
- Musical morphisms: $\hat{\Omega}_j : \mathfrak{X}^j \longrightarrow \Lambda^{k-j}$.

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Multisymplectic examples

Example ($\Lambda^k Q$ as a multisymplectic manifold)

Let $M = \Lambda^k Q$ and $\Omega = -\mathrm{d}\Theta$, where Θ is the Liouville k-form

$$\Theta_\alpha(X_1, \ldots, X_k) := \alpha(\nu_* X_1, \ldots, \nu_* X_k).$$

In coordinates,

$$\Theta = p_{i_1,\ldots,i_k} \mathrm{d}q^{i_1} \wedge \cdots \wedge \mathrm{d}q^{i_k}.$$

Then $(\Lambda^k Q, \Omega)$ is a multisymplectic manifold and

$$\Omega = -\mathrm{d}p_{i_1,\ldots,i_k} \wedge \mathrm{d}q^{i_1} \wedge \cdots \wedge \mathrm{d}q^{i_k}.$$

Example ($\Lambda^k_r Q$ as a multisymplectic manifold)

$\Lambda^k_r Q$ is the set of the r-horizontal k-forms (with respect to a fibration $\pi : Q \longrightarrow P$). Let Ω_r be the restriction of Ω. Then $(\Lambda^k_r Q, \Omega_r)$ is a multisymplectic manifold.

The Multi-Index Notation

Usually, given a function $f : \mathbb{R}^m \longrightarrow \mathbb{R}$, its partial derivatives are denoted by

$$f_{i_1 i_2 \cdots i_k} = \frac{\partial^k f}{\partial x^{(i_1, i_2, \dots, i_k)}} := \frac{\partial^r f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}}.$$

What matters is not the order in which the derivatives are taken, but the number of times with respect to each variable.

Definition

- A multi-index is a m-tuple $I \in \mathbb{N}^m$.
- The i-th component of I is denoted $I(i)$.
- Addition and "substraction" of multi-indexes are defined componentwise, $(I \pm J)(i) = I(i) \pm J(i)$.
- The length of I is the sum $|I| = \sum_i I(i)$, and its factorial $I! = \Pi_i I(i)!$.
- $1_i = (\delta_j^i) = (0, \dots, 1, \dots, 0)$.

The Multi-Index Notation

We will denote the partial derivatives of a function $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ by

$$f_I = \frac{\partial^{|I|} f}{\partial x^I} := \frac{\partial^{I(1)+I(2)+\cdots+I(m)} f}{\partial x_1^{I(1)} \partial x_2^{I(2)} \cdots \partial x_m^{I(m)}}.$$

Given $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$,

- $\frac{\partial^3 f}{\partial x_1 \partial x_2 \partial x_1}$ will be $f_{(2,1,0)}$ instead of f_{121}.
- $\frac{\partial}{\partial x_1} \frac{\partial^2 f}{\partial x_2 \partial x_1}$ will be $f_{(1,1,0)+1_1}$ (or $f_{(2,0,0)+1_2}$).

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The Multi-Index Notation

Lemma

- $$\sum_{I+1_i=J} \frac{I(i)+1}{|I|+1} = 1$$
- $$\sum_{|I|=l-1} \sum_{i=0}^{m} a_{I,i} = \sum_{|J|=l} \sum_{I+1_i=J} a_{I,i}$$
- $$\sum_{|J|=l} b^J a_J = \sum_{|I|=l-1} \sum_{i=1}^{m} \frac{I(i)+1}{|I|+1} (b^{I+1_i} + Q^{I,i}) a_{I+1_i} \text{ where}$$
  $$\sum_{I+1_i=J} \frac{I(i)+1}{|I|+1} Q^{I,i} = 0$$

Definition

- (E, π, M) will be a fiber bundle ($\dim M = m$ and $\dim E = m + n$).
- Adapted coordinated systems will be of the form (x^i, u^α).
- $\phi, \psi \in \Gamma(E)$ are k-equivalent at p if for all α, i_j and r

$$\phi(p) = \psi(p) \text{ and } \left.\frac{\partial^r \phi^\alpha}{\partial x^{i_1} \cdots \partial x^{i_r}}\right|_p = \left.\frac{\partial^r \psi^\alpha}{\partial x^{i_1} \cdots \partial x^{i_r}}\right|_p .$$

Definition

- The *k-jet of ϕ at p* is $j^k_p \phi := [\phi]^k_p$.
- The *k-th jet manifold of π* is $J^k\pi := \{j^k_p\phi \ : \ p \in M, \phi \in \Gamma_p(\pi)\}$.

Adapted coordinates (x^i, u^α) on E induce adapted coordinates (x^i, u^α_I) on $J^k\pi$ (with $0 \leq |I| \leq k$):

$$u^\alpha_I(j^k_p\phi) = \left.\frac{\partial^{|I|}\phi^\alpha}{\partial x^I}\right|_p .$$

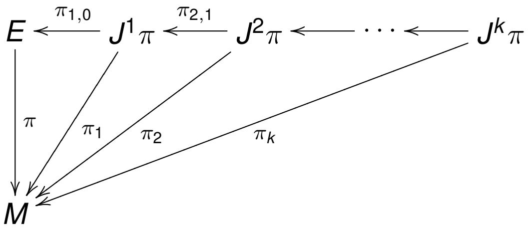
Properties

- $(J^k\pi, \pi_k, M)$ is a fiber bundle, where

$$\pi_k(j^k_p\phi) = p \quad \text{(in coordinates } \pi_k(x^i, u^\alpha_I) = (x^i)\text{)}.$$

- $J^k\pi$ fibers over the l-jet manifolds $J^l\pi$ ($l \leq k$), where

$$\pi_{k,l}(j^k_p\phi) = j^l_p\phi \quad \text{(in coord. } \pi_{k,l}(x^i, u^\alpha_I) = (x^i, u^\alpha_J), |I| \leq k, |J| \leq l\text{)}.$$



- $(J^k\pi, \pi_{k,k-1}, J^{k-1}\pi)$ is an affine fiber bundle.

The (extended) dual affine bundle

Definition

- The *extended dual affine bundle* $(J^k\pi^\dagger, \pi^\dagger_{k,k-1}, J^{k-1}\pi)$ is a fiber bundle whose fibers consist of affine maps of the corresponding fibers of the affine bundle $(J^k\pi, \pi_{k,k-1}, J^{k-1}\pi)$.
- The *dual affine bundle* $(J^k\pi^*, \pi^*_{k,k-1}, J^{k-1}\pi)$ is a fiber bundle whose fibers consist of classes of affine maps of the corresponding fibers of the affine bundle $(J^k\pi, \pi_{k,k-1}, J^{k-1}\pi)$, which differ by a constant.

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The (extended) dual affine bundle

Definition

- The *extended dual affine bundle* $(J^k\pi^\dagger, \pi^\dagger_{k,k-1}, J^{k-1}\pi)$.
- The *dual affine bundle* $(J^k\pi^*, \pi^*_{k,k-1}, J^{k-1}\pi)$.

- There exist canonical isomorphisms $J^k\pi^\dagger \approx \Lambda^m_2(J^{k-1}\pi)$ and $J^k\pi^* \approx \Lambda^m_2(J^{k-1}\pi)/\Lambda^m_1(J^{k-1}\pi)$.
- If $\omega \in \Lambda^m_2(J^{k-1}\pi)$, then $\omega = p\,\mathrm{d}^m x + p^{I,i}_\alpha \mathrm{d}u^\alpha_I \wedge \mathrm{d}^{m-1}x_i$, $|I| \leq k-1$. Thus we have coordinates $(x^i, u^\alpha_I, p, p^{I,i}_\alpha)$ on $\Lambda^m_2(J^{k-1}\pi)$.
- There is a natural pairing between $J^k\pi$ and $J^k\pi^\dagger$, $\Phi(j^k_x\phi, \omega_{j^{k-1}_x\phi}) := a(x)$, sucht that $a(x)\eta(x) = (j^{k-1}\phi)^*\omega_{j^{k-1}_x\phi}$. In coordinates: $\Phi(x^i, u^\alpha_I, u^\alpha_K, p^{I,i}_\alpha, p) = p^{I,i}_\alpha u^\alpha_{I+1_i} + p$.
- $\Lambda^m_2(J^{k-1}\pi)$ has a canonical multisymplectic structure,

$$\Omega = -\mathrm{d}p \wedge \mathrm{d}^m x - \mathrm{d}p^{I,i}_\alpha \wedge \mathrm{d}u^\alpha_I \wedge \mathrm{d}^{m-1}x_i.$$

Mechanics (the multisymplectic case)

1. The Lagrangian $L : J^1\pi \longrightarrow \mathbb{R}$.
2. Let $\hat{p}^i_\alpha = \frac{\partial L}{\partial u^\alpha_i}$ and $\hat{p} = L - u^\alpha_i \hat{p}^i_\alpha$.
3. The Poincaré-Cartan m-form $\Theta_L := \hat{p}^i_\alpha \mathrm{d}u^\alpha \wedge \mathrm{d}^{m-1}x_i + \hat{p}\mathrm{d}^m x$.
4. The Poincaré-Cartan $(m+1)$-form $\Omega_L := -\mathrm{d}\Theta_L$.
5. $(J^1\pi, \Omega_L)$ is multisymplectic whenever L is regular.

Theorem

A section $\phi : M \longrightarrow E$ satisfies the Euler-Lagrange equations

$$(j^1\phi)^* \left(\frac{\partial L}{\partial u^\alpha} - \frac{\mathrm{d}}{\mathrm{d}x^i} \frac{\partial L}{\partial u^\alpha_i} \right) = 0$$

iff $\forall X \in \mathfrak{X}(J^1\pi)$

$$(j^1\phi)^*(i_X \Omega_L) = 0.$$

Definition

A section $\sigma : M \longrightarrow J^1\pi$ satisfies the *DeDonder equations* iff

$$(\sigma)^*(i_X \Omega_L) = 0, \quad \forall X \in \mathfrak{X}(J^1\pi).$$

Theorem

Let Γ be a connection in $\pi_1 : J^1\pi \longrightarrow M$ whose horizontal projector **h** *satisfies*

$$i_{\mathbf{h}}\Omega_L = (m-1)\Omega_L.$$

The integral sections of Γ satisfies the DeDonder equations.

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First order Skinner-Rusk formalism

1. Consider the fibered product $W_0 := J^1\pi \times_E \Lambda_2^m E$.
2. The pairing $\Phi(x^i, u^\alpha, u_i^\alpha, p_\alpha^i, p) = p_\alpha^i u_i^\alpha + p$.
3. The dynamical function $H := \Phi - L \circ pr_1$.
4. The canonical multisymplectic $(m+1)$-form $\Omega = -\mathrm{d}p \wedge \mathrm{d}^m x - \mathrm{d}p_\alpha^i \wedge \mathrm{d}u^\alpha \wedge \mathrm{d}^{m-1}x_i$.
5. The $(m+1)$-form $\Omega_0 := \Omega + \mathrm{d}H \wedge \mu$.
6. We search an Ehresmann connection in the fibered bundle $\pi_{W_0,M} : W_0 \longrightarrow M$ whose horizontal projector is the solution of

$$i_{\mathbf{h}}\Omega_0 = (m-1)\Omega_0 .$$

Higher order Skinner-Rusk formalism

1. Consider the fibered product $W_0 := J^k\pi \times_{J^{k-1}\pi} \Lambda_2^m(J^{k-1}\pi)$.
2. The pairing $\Phi(x^i, u_I^\alpha, u_K^\alpha, p_\alpha^{I,i}, p) = p_\alpha^{I,i} u_{I+1_i}^\alpha + p$.
3. The dynamical function $H := \Phi - L \circ pr_1$.
4. The canonical multisymplectic $(m+1)$-form $\Omega = -\mathrm{d}p \wedge \mathrm{d}^m x - \mathrm{d}p_\alpha^{I,i} \wedge \mathrm{d}u_I^\alpha \wedge \mathrm{d}^{m-1} x_i$.
5. The $(m+1)$-form $\Omega_0 := \Omega + \mathrm{d}H \wedge \mu$.
6. We search an Ehresmann connection in the fibered bundle $\pi_{W_0,M} : W_0 \longrightarrow M$ whose horizontal projector is the solution of

$$i_{\mathbf{h}}\Omega_0 = (m-1)\Omega_0.$$

Solving the Skinner-Rusk equation

We first have to restrict to the space where solutions exist:

$$\begin{aligned} W_1 \quad := \quad & \{ w \in W_0 \ / \ \exists \mathbf{h}_w : T_w W_0 \longrightarrow T_w W_0 \text{ linear s.t. } \mathbf{h}_w^2 = \mathbf{h}_w, \\ & \ker \mathbf{h}_w = (V\pi_{W_0,M})_w, i_{\mathbf{h}_w}\Omega_{H_0}(w) = (m-1)\Omega_{H_0}(w) \}. \end{aligned}$$

The projectors must have the form

$$\mathbf{h}_w = \left(\frac{\partial}{\partial x^i} + A^\alpha_{Ji} \frac{\partial}{\partial u^\alpha_J} + B^{Ij}_{\alpha i} \frac{\partial}{\partial p^{I,j}_\alpha} + C_i \frac{\partial}{\partial p} \right) \otimes \mathrm{d}x^i .$$

We start to compute...

Solving the Skinner-Rusk equation

$$
\begin{aligned}
& i_{\mathbf{h}_w}\Omega_0 - (m-1)\Omega_0 = \\
&= \left(B^{Ii}_{\alpha i} \mathrm{d}u^\alpha_I - A^\alpha_{Ii} \mathrm{d}p^{I,i}_\alpha + p^{I,i}_\alpha \mathrm{d}u^\alpha_{I+1_i} + u^\alpha_{I+1_i} \mathrm{d}p^{I,i}_\alpha - \frac{\partial L}{\partial u^\alpha_J} \mathrm{d}u^\alpha_J \right) \wedge \mathrm{d}^m x \\
&= \Bigg[\left(B^{\ i}_{\alpha i} - \frac{\partial L}{\partial u^\alpha} \right) \mathrm{d}u^\alpha + \sum_{|I'|=1}^{k-1} \left(B^{I'i}_{\alpha i} - \frac{\partial L}{\partial u^\alpha_{I'}} \right) \mathrm{d}u^\alpha_{I'} + \sum_{J'=0}^{k-2} p^{I,i}_\alpha \mathrm{d}u^\alpha_{I+1_i} \\
&\quad + \sum_{|K|=k} -\frac{\partial L}{\partial u^\alpha_K} \mathrm{d}u^\alpha_K + \sum_{K'=k-1} p^{K',i}_\alpha \mathrm{d}u^\alpha_{K'+1_i} \\
&\quad + \sum_{|I|=0}^{k-1} \left(u^\alpha_{I+1_i} - A^\alpha_{Ii} \right) \mathrm{d}p^{I,i}_\alpha \Bigg] \wedge \mathrm{d}^m x
\end{aligned}
$$

Equating to zero...

Solving the Skinner-Rusk equation

$$
\begin{aligned}
B_{\alpha j}^{\ j} &= \frac{\partial L}{\partial u^\alpha}; \\
p_\alpha^{I,i} &= \frac{I(i)+1}{|I|+1}\left(\frac{\partial L}{\partial u^\alpha_{I+1_i}} - B_{\alpha j}^{I+1_i j} + Q_\alpha^{Ii}\right), \text{ with } |I| = 0,\ldots,k-2; \\
p_\alpha^{I,i} &= \frac{I(i)+1}{|I|+1}\left(\frac{\partial L}{\partial u^\alpha_{I+1_i}} + Q_\alpha^{Ii}\right), \text{ with } |I| = k-1; \\
A_{Ii}^\alpha &= u^\alpha_{I+1_i}, \text{ with } |I| = 0,\ldots,k-1;
\end{aligned}
$$

Getting rid of the Q's...

Solving the Skinner-Rusk equation

We finally obtain:

$$B_{\alpha j}^{\ j} = \frac{\partial L}{\partial u^\alpha}; \tag{1}$$

$$\sum_{I+1_j=J} p_\alpha^{I,i} = \frac{\partial L}{\partial u_J^\alpha} - B_{\alpha j}^{Jj}, \text{ with } |I| = 1, \dots, k-1; \tag{2}$$

$$\sum_{I+1_i=K} p_\alpha^{I,i} = \frac{\partial L}{\partial u_K^\alpha}, \text{ with } |K| = k; \tag{3}$$

$$A_{Ii}^\alpha = u_{I+1_i}^\alpha, \text{ with } |I| = 0, \dots, k-1.$$

Regularity, $\mathbf{h}_w(T_w W_0) \subset i_*(TW_2)\ \forall w \in W_2$,

$$\sum_{I+1_j=K} B_{\alpha j}^{Ij} = \frac{\partial^2 L}{\partial x^j \partial u_K^\alpha} + u_{I+1_j}^\beta \frac{\partial^2 L}{\partial u_I^\beta \partial u_K^\alpha} + A_{K'j}^{\alpha'} \frac{\partial^2 L}{\partial u_{K'}^{\alpha'} \partial u_K^\alpha}, \text{ with } |K| = k. \tag{4}$$

The first order case (k=1)

Main equations

$$B_{\alpha j}^{\ j} = \frac{\partial L}{\partial u^\alpha};$$
$$p_\alpha^i = \frac{\partial L}{\partial u_i^\alpha}.$$

Regularity condition

$$B_{\alpha j}^{\ i} = \frac{\partial^2 L}{\partial x^j \partial u_i^\alpha} + u_j^\beta \frac{\partial^2 L}{\partial u^\beta \partial u_i^\alpha} + A_{i'j}^{\alpha'} \frac{\partial^2 L}{\partial u_{i'}^{\alpha'} \partial u_i^\alpha}.$$

The one-dimensional case ($m = 1$)

Main equations

$$\begin{aligned} B_\alpha &= \frac{\partial L}{\partial u^\alpha}; \\ p_\alpha^j &= \frac{\partial L}{\partial u_j^\alpha} - B_\alpha^{j+1}, \text{ with } j = 1, \ldots, k-1; \\ p_\alpha^k &= \frac{\partial L}{\partial u_k^\alpha}. \end{aligned}$$

Regularity condition

$$B_\alpha^k = \frac{\partial^2 L}{\partial x \partial u_k^\alpha} + u_{i+1}^\beta \frac{\partial^2 L}{\partial u_i^\beta \partial u_k^\alpha} + A_{k'}^{\alpha'} \frac{\partial^2 L}{\partial u_{k'}^{\alpha'} \partial u_k^\alpha}.$$

The multisymplectic phase space

$$W_2 := \{w \in W_1 \ : \ H(w) = 0\} = \{w \in W_0 \ : \ (3) \text{ and } H(w) = 0\}$$

Theorem

(W_2, Ω_2) is multisymplectic iff L is regular ($\det\left(\frac{\partial^2 L}{\partial u_K^\alpha \partial u_{K'}^{\alpha'}}\right) \neq 0$*).*

What now?

- Higher-order Euler-Lagrange equations.
- Higher-order Gotay-Nester-Hinds algorithm.
- Higher-order control theory (María Barbero).
- Higher-order paella.

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The multisymplectic phase space

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Theorem

(W_2, Ω_2) is multisymplectic iff L is regular ($\det \left(\frac{\partial^2 L}{\partial u^\alpha_K \partial u^{\alpha'}_{K'}} \right) \neq 0$).

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The multisymplectic phase space

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Theorem

(W_2, Ω_2) *is multisymplectic iff L is regular* ($\det \left(\frac{\partial^2 L}{\partial u_K^\alpha \partial u_{K'}^{\alpha'}} \right) \neq 0$).

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Theorem

(W_2, Ω_2) is multisymplectic iff L is regular $\left(\det\left(\frac{\partial^2 L}{\partial u_K^\alpha \partial u_{K'}^{\alpha'}}\right) \neq 0\right)$*.*

What now?

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- Higher-order paella.

Higher-order Euler-Lagrange equations

Given an Ehresmann connection on $\pi_{W_2 M} : W_2 \longrightarrow M$, whose horizontal projector $\mathbf{h}$ is a solution of the Skinner-Rusk equation

$$i_{\mathbf{h}} \Omega_2 = (m-1)\Omega_2,$$

consider the "horizontal projector" $\bar{\mathbf{h}} = i_* \mathbf{h}\, i_*^{-1}$ defined on $i_*(TW_2)$.

Theorem

Let σ be a section of $\pi_{W_2 M} : W_2 \longrightarrow M$ and denote $\bar{\sigma} = i \circ \sigma$. If σ is an integral section of $\mathbf{h}$, then σ is holonomic,

$$pr_1 \circ \bar{\sigma} = j^k(\pi_{W_2 M} \circ \sigma),$$

and satisfies the higher-order Euler-Lagrange equations:

$$j^{2k}(\pi_{W_2 M} \circ \sigma)^* \left(\sum_{|J|=0}^{k} (-1)^{|J|} \frac{\mathrm{d}^{|J|}}{\mathrm{d}x^J} \frac{\partial L}{\partial u^\alpha_J} \right) = 0.$$

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The End

- All that has a beginning has an end.
(The Oracle, *The Matrix Trilogy*)