Quasi-Lie schemes: theory and applications

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Outline

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   - Lie Systems and superposition rules.
   - Lie's theorem, problems and solutions.
2. **Quasi-Lie schemes and quasi-Lie systems**
   - Quasi-Lie schemes.
   - The main theorem of the theory of quasi-Lie schemes.
   - Quasi-Lie systems and time-dependent superposition rules.
3. **Examples and applications**
   - The second order Riccati equation.
   - Other applications.
   - Outlook.

Differential equations to study.

Geometrical description

Suppose a differential equation on $\mathbb{R}^n$

$$\frac{dx^i}{dt} = Y^i(t,x), \qquad i = 1, \ldots, n,$$

From a differential eometric point of view, this is the equation of the integral curves for the time-dependent vector field

$$Y(t,x) = \sum_{i=1}^{n} Y^i(t,x) \frac{\partial}{\partial x^i}.$$

Superposition Rules.

Definition

The initial differential equation admits a superposition rule if there exists a map $\Phi : \mathbb{R}^{n(m+1)} \to \mathbb{R}^n$ such that its general solution $x(t)$ can be written as

$$x(t) = \Phi(x_{(1)}(t), \ldots, x_{(m)}(t), k_1, \ldots, k_n).$$

with $\{x_{(1)}(t), \ldots, x_{(m)}\}$ a family of particular solutions and $\{k_1, \ldots, k_n\}$ a set of constants.

Examples

- Linear inhomogeneous differential equations.
- Riccati equations.

Definition of Lie system.

Definition of Lie system

The initial differential equation is called a Lie system if there exist vector fields $X_{(1)},\ldots,X_{(r)}$ on $\mathbb{R}^n$ such that

- The time-dependent vector field $Y(t,x)$ verifies

$$Y(t,x)=\sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha(x).$$

- The vector fields $\{X_{(\alpha)}\,|\,\alpha=1,\ldots,r\}$ verify

$$\left[X_{(\alpha)},X_\beta\right]=c_{\alpha\beta}^{\gamma}X_{(\gamma)}$$

for certain r^3 constants $c_{\alpha\beta}^{\gamma}$.

Lie’s theorem and practical problems.

Lie’s Theorem

A first-order differential equation admits a superposition rule if and only if it is a Lie system.

PROBLEMS

1. Generally, it is difficoult checking out that a differential equation is a Lie system and even more finding a superposition rule.
2. We maybe do not know if a differential equation admits any, one or more superposition rules.
3. There exist many important differential equations which are Lie systems, but in general, these cases are rare.

Solutions for the problems of Lie's theorem.

Provided solutions

- Quasi-Lie schemes and quasi-Lie systems.
- Generalized Lie theorem.

Fundamentals.

Denifition of quasi-Lie scheme

A quasi-Lie scheme $\mathfrak{sc}(W, V)$ is given by:

- A non-null Lie algebra of vector fields W.
- A linear space V spanned by a finite set of vector fields.

These two elements verify:

- $W \subset V$.
- $[W, V] \subset V$.

A scheme for the Abel equation.

Example

Define on $\mathbb{R}$ the set of vector fields

$$X_0=\frac{\partial}{\partial x},\quad X_1=x\frac{\partial}{\partial x},\quad X_2=x^2\frac{\partial}{\partial x},\quad X_3=x^3\frac{\partial}{\partial x}\,.$$

and

$$W_{Abel}=\langle X_0,X_1\rangle,\qquad V_{Abel}=\langle X_0,X_1,X_2,X_3\rangle.$$

The linear space $W_{Abel}\subset V_{Abel}$ is a Lie algebra because $[X_0,X_1]=X_1$. Moreover, as

$$\begin{array}{ll}[X_0,X_2]=2X_1, & [X_0,X_3]=3X_2,\\ [X_1,X_2]=X_2, & [X_1,X_4]=2X_3,\end{array}$$

then $[W_{Abel},V_{Abel}]\subset V_{Abel}$.

Differential equations described by a quasi-Lie scheme.

Definition

A differential equation described by means of a time-dependent vector field $Y(t,x)$ can be described by a scheme $\mathfrak{sc}(W,V)$ if

$$Y(t,x)=\sum_{\alpha=1}^{s} b_\alpha(t)X_\alpha(x)$$

with $\{X_\alpha\,|\,\alpha=1,\ldots,s\}$ a basis which spans V.

Theorem

Every Lie system can be described through a quasi-Lie scheme.

Example

The Abel equation

$$\dot{x} = f_3(t)x^3 + f_2(t)x^2 + f_1(t)x + f_0(t),$$

describes the integrals for the time-dependent vector field

$$X(t) = f_3(t)X_3 + f_2(t)X_2 + f_1(t)X_1 + f_0(t)X_0\,.$$

Hence, it can be described by the quasi-Lie scheme $\mathfrak{sc}(W_{Abel}, V_{Abel})$.

Flows for a quasi-Lie scheme.

Generalized flows of a quasi-Lie scheme

Given any time-dependent vector field X given by

$$X(t) = \sum_{\alpha=1}^{s} b_\alpha(t) X_\alpha$$

with $\{X_\alpha \mid \alpha = 1, \ldots, s\}$ a basis for W. This vector field can be considered as a curve $b(t) = (b_1(t), \ldots, b_s(t))$. Then, there exists a generalized flow for $X(t)$ which we denote

$$g_t(b_1(t), \ldots, b_s(t)).$$

Let us denote this family of flows as $\mathfrak{fl}(W)$.

Actions of flows.

Time-dependent vector fields and flows

Any time-dependent vector field admits a genealized flow g_t^X. The knowledge of a time-dependent vector field is equivalent to the knowledge of its flow.

Action of flows

Given a time-dependent vector field $Y(t,x)$ with generalized flow g_t^Y and any other generalized flow g_t, we define the action of g_t on Y, denoted by $(g_t)_\star Y = Y'$, with Y' the time-dependent vector field with generalized flow $g_t \circ g_t^Y$.

The main theorem of the theory of quasi-Lie schemes.

Main theorem of the theory of quasi-Lie schemes

Suppose a quasi-Lie scheme $\mathfrak{sc}(W, V)$. Given a generalized flow $g_t \in \mathfrak{fl}(W)$ and a time-dependent vector field

$$Y(t,x) = \sum_{\alpha=1}^{s} b_\alpha(t) X_\alpha,$$

with $\{X_\alpha \,|\, \alpha = 1, \ldots, s\}$ a basis for V, then

$$(g_t)_\star Y = \sum_{\alpha=1}^{s} b'_\alpha(t) X_\alpha.$$

Time-dependent superposition rules and quasi-Lie systems.

Definition of quasi-Lie systems

We say that a time-dependent vector field $Y(t,x)$ is a quasi-Lie system respecto to a quasi-Lie scheme $\mathfrak{sc}(W,V)$ if

1. Y can be considered as a curve in V.
2. There exists an element $g_t \in \mathfrak{fl}(W)$ such that $(g_t)_\star Y$ is a Lie system.

Main property of quasi-Lie systems

If Y is a quasi-Lie system respect a quasi-Lie scheme it can be shown that its general solution can be written as

$$x(t) = \Psi(t, x_{(1)}(t), \ldots, x_{(m)}(t); k_1, \ldots, k_n),$$

with $\{x_{(1)}(t), \ldots, x_{(m)}(t)\}$ a family of particular solutions and $\{k_1, \ldots, k_n\}$ a set of constants.

A second order Riccati equation.

Problem

Let us study the second-order Riccati equation

$$\ddot{x}+(b_0(t)+b_1(t)x)\dot{x}+a_0(t)+a_1(t)x+a_2(t)x^2+a_3(t)x^3=0,$$

with

$$b_1(t)=3\sqrt{a_3(t)},\qquad b_0(t)=\frac{a_2(t)}{\sqrt{a_3(t)}}-\frac{\dot{a}_3(t)}{2a_3(t)}.$$

By means of the change of variables $\dot{x}=v$, we transform the latter second-order differential equation into the first-order one

$$\begin{cases}\dot{x}=v,\\ \dot{v}=-(b_0(t)+b_1(t)x)v-a_0(t)-a_1(t)x-a_2(t)x^2-a_3(t)x^3.\end{cases}$$

A quasi-Lie scheme for the second-order Riccati equation.

Example

Define on $\mathrm{T}\mathbb{R}$ the set of vector fields

$$\begin{aligned} Y_1 &= v\frac{\partial}{\partial x}, & Y_2 &= v\frac{\partial}{\partial v}, & Y_3 &= xv\frac{\partial}{\partial v}, & Y_4 &= \frac{\partial}{\partial v}, \\ Y_5 &= x\frac{\partial}{\partial v}, & Y_6 &= x^2\frac{\partial}{\partial v}, & Y_7 &= x^3\frac{\partial}{\partial v}, & Y_8 &= x\frac{\partial}{\partial x}. \end{aligned}$$

Then, we construct W_{Ricc} and V_{Ricc} as $W_{Ricc} = \langle Y_2, Y_8\rangle$, $V_{Ricc} = \langle Y_1, \ldots, Y_8\rangle$. Evidently $W_{Ricc} \subset V_{Ricc}$ is an abelian two dimensional Lie algebra and as

$$\begin{aligned} &[Y_2, Y_1] = Y_1, && [Y_2, Y_3] = 0, && [Y_2, Y_4] = -Y_2, \\ &[Y_2, Y_5] = -Y_5, && [Y_2, Y_6] = -Y_6, && [Y_2, Y_7] = -Y_7, \\ &[Y_8, Y_1] = -Y_1, && [Y_8, Y_3] = Y_3, && [Y_8, Y_4] = 0, \\ &[Y_8, Y_5] = Y_5, && [Y_8, Y_6] = 2Y_6, && [Y_8, Y_7] = 3Y_7. \end{aligned}$$

then $[W_{Ricc}, V_{Ricc}] \subset V_{Ricc}$.

Flows of the quasi-Lie scheme for the second-order Riccati equation.

The second-Riccati equation can be understood as the differential equation of the integral curves for the time-dependent vector field given by

$$X(t) = Y_1 - b_0(t)Y_2 - b_1(t)Y_3 - a_0(t)Y_4 - a_1(t)Y_5 - a_2(t)Y_6 - a_3(t)Y_7.$$

So, the quasi-Lie scheme $\mathfrak{sc}(W_{Ricc}, V_{Ricc})$ allows us to deal with such an equation.

Flow of the quasi-Lie scheme

The set $\mathfrak{fl}(W_{Ricc})$ is given by

$$\begin{aligned} x(t) &= \alpha(t)x'(t), \\ v(t) &= \beta(t)v'(t). \end{aligned}$$

Applications of the main theorem of the theory of quasi-Lie schemes.

Consecuence of our main theorem.

The flows of our scheme transforms the initial second-order Riccati equation into

$$
\begin{aligned}
\frac{dx'}{dt} &= \frac{\beta(t)}{\alpha(t)}v' - \frac{\dot\alpha(t)}{\alpha(t)}x',\\
\frac{dv'}{dt} &= -\frac{a_0(t)}{\beta(t)} + \left(-b_1(t)\alpha(t)v' - \frac{\alpha(t)a_1(t)}{\beta(t)}\right)x' - \frac{a_2(t)\alpha^2(t)}{\beta(t)}x'^2\\
&\quad - \frac{a_3(t)\alpha^3(t)}{\beta(t)}x'^3 - \frac{b_0(t)\beta(t) + \beta'(t)}{\beta(t)}v'.
\end{aligned}
$$

The second order Riccati equation.

Fix $b_3(t)=\sqrt{a_3(t)}$ and $\alpha(t)=1$. We obtain that

$$\left\{\begin{aligned} \frac{dx'}{dt}&=\sqrt{a_3(t)}v',\\ \frac{dv'}{dt}&=-\frac{a_0(t)}{\sqrt{a_3(t)}}-\sqrt{a_3(t)}(3v'x'+x'^3)-\frac{a_1(t)}{\sqrt{a_3(t)}}x'\\ &\quad-\frac{a_2(t)}{\sqrt{a_3(t)}}(v'+x'^2). \end{aligned}\right.$$

But this is the system of differential equations related to the time-dependent vector field

$$X(t)=\sqrt{a_3(t)}X_1-\frac{a_0(t)}{\sqrt{a_3(t)}}X_2-\frac{a_1(t)}{\sqrt{2a_3(t)}}(X_3+X_7)-\frac{a_2(t)}{\sqrt{a_3(t)}}X_0$$

And this is a Lie system related to a set of vector fields which span a $\mathfrak{sl}(3,\mathbb{R})$ Lie algebra.

Time-dependent superposition rules.

The time-dependent superposition rule for the second-order Riccati equation reads

$$x_5=\frac{-x_4F_{123}+G_{1234}\Lambda_1+G_{1243}\Lambda_3-x_3F_{124}\Lambda_1\Lambda_3}{-F_{123}+(F_{124}-F_{324})\Lambda_3+(F_{123}-F_{423})\Lambda_1-\Lambda_1\Lambda_3F_{124}},$$

where

$$\begin{aligned}G_{abcd}=&x_a((v_d-v_c)x_b+(v_b-v_d)x_c+(x_b-x_c)x_bx_c+(x_c-x_b)x_ax_d)+\\&x_d((v_c-v_a)x_b+(v_a-v_b)x_c+(x_c-x_b)x_bx_c+(x_b-x_c)x_ax_d)\end{aligned}$$

and

$$F_{abc}=v_a(x_c-x_b)+v_b(x_a-x_c)+v_c(x_b-x_a)+(x_a-x_b)(x_b-x_c)(x_c-x_a).$$

Other applications.

1. Abel equations can be described by a quasi-Lie scheme. Those Abel equations which are quasi-Lie systems seem to be related to the so-called integrable cases.
2. Dissipative Ermakov systems can be studied through a quasi-Lie schemes. Time-dependent superposition rules have been obtained.
3. Emden-Fowler equations can be described by a quasi-Lie schemes. This method can be used to obtain exact solutions, i.e. the Lane-Emden equation can be solved analytically.
4. Non-linear oscillators have been studied by means of this method. Integrals of motion and other properties have been obtained.
5. This method is related to a generalized Lie theorem which allows to describe when a differential equations admits a time-dependent superposition rule.

Outlook

1. There are many differential equations that can be studied by means of quasi-Lie schemes. This method allows to obtain properties for them but it has not been applied yet to many cases.
2. Still it has to be proved when there is a way to know if a differential equation is not a quasi-Lie system.
3. Extensions of quasi-Lie schemes to deal with PDE's and Quantum Mechanics have partially been done.
4. Applications on many physical problems in Quantum Mechanics.

THE END

THANK FOR YOUR ATTENTION