The Marle-Guillemin-Sternberg form. What is it and how to use it.

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Abstract

We give a basic exposition of the construction of the Marle-Guillemin-Sternberg form, focusing on its applications to problems involving Hamiltonian Lie group actions in symplectic geometry and Hamiltonian dynamics.

Introduction

The problem of finding normal forms for geometric structures on a given manifold is about providing local models for the manifold on which the geometric structure in question has a particular “nice” expression (meaning that it is commonly accepted as convenient or simpler than the original one). The power of using normal forms is that usually this nice expression simplifies some problem of local nature under study that involves the geometric structure in question. Some examples of local models and normal forms in differential geometry are the following:

Charts. For any smooth manifold M and $x \in M$, the existence of a smooth structure on M guarantees that there is a local diffeomorphism $\phi : M \to \mathbb{R}^n$ defined on a neighborhood of any point $x \in M$ such that $\phi(x) = 0$. We say then that $\mathbb{R}^n$ is a local model for M near x. Since in this case there is no additional geometric structure on M, there is no normal form.

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Morse Lemma. If in addition, f is a smooth function defined on M such that x_0 is a non-degenerate critical point, the Morse Lemma guarantees that there are special charts centered at x_0, with coordinates $(x^1, \ldots, x^n)$ such that

$$f(x^1, \ldots, x^n) = f(x_0) + (x^1)^2 + \ldots + (x^k)^2 - (x^{k+1})^2 - \ldots - (x^n)^2, \qquad (1)$$

where k is the number of positive eigenvalues of the Hessian of f. In this example, f is the geometric structure on M, and the Morse Lemma provides a local model for M near x (given by $\mathbb{R}^n$) together with a normal form for f (given by formula (1)). See [17] for more on this topic.

Koszul-Palais Tube. Let G be a Lie group acting on M in a proper fashion (the action is proper if the map $G \times M \to M \times M$; $(g, x) \mapsto (x, g \cdot x)$ is proper) and let $G \cdot x$ denote the orbit through x, $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{g} \cdot x = T_x(G \cdot x)$. Let $G_x := \{g \in G \,:\, g \cdot x = x\}$ (the isotropy group of x, it is a compact subgroup of G). Recall that the lifted action of G to TM induces a linear action on T_xM. Let S be a G_x-invariant subspace defined by a G_x-invariant splitting

$$T_xM = \mathfrak{g} \cdot x \oplus S.$$

Since G_x is compact such an splitting always exists. S is called a slice at x.

There is a (free) action of G_x on $G \times S$ given by

$$h \cdot (g, s) = (gh^{-1}, h \cdot s) \quad \forall h \in G_x,\ g \in G,\ s \in S.$$

The quotient space of this action will be denoted by $G \times_{G_x} S$ and its elements are equivalence classes $[g, s]$, where (g, s) and (g', s') are equivalent iff they belong to the same G_x-orbit. Note that there is a well defined action of G on $G \times_{G_x} S$ given by

$$g' \cdot [g, s] = [g'g, s]. \qquad (2)$$

The Koszul-Palais Tube Theorem says that there is a local diffeomorphism $\phi : G \times_{G_x} S \to M$ satisfying

(i) $\phi([e, 0]) = x$,

(ii) $\phi(G \times_{G_x} \{0\}) = G \cdot x$, and

(iii) ϕ is G-equivariant with respect to the action (2) on $G \times_{G_x} S$ and the original action on M.

The map ϕ is called a tube.

That is, the Koszul-Palais Tube Theorem gives a local model for a smooth manifold acted upon properly by a Lie group near a group orbit $G \cdot x$ (the space $G \times_{G_x} S$) and a normal form for the action (formula (2)). Note that we can locally model our manifold near objects other than a point, and that then the local model does not have to be a particular case of a chart. This result originally appeared in [6, 22].

Note: When we say that the map ϕ above is a local diffeomorphism, we mean specifically the following: There is a G_x-invariant open neighborhood $0 \in U_S$ in S, and a G-invariant neighborhood $G \cdot x \subset U_M$ in M such that

$$\phi : G \times_{G_x} U_S \longrightarrow U_M$$

is a diffeomorphism satisfying the properties (i) to (iii).

Darboux Theorem. Let now ω be a symplectic form on M (ω is a non-degenerate differential two-form and $d\omega = 0$). We are now assuming that M is $2n$-dimensional. This form ω is the geometric structure on M. In this case the Darboux Theorem guarantees that there are special charts centered at x with coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ for on which

$$\omega = dx^1 \wedge dy^1 + \ldots + dx^n \wedge dy^n. \tag{3}$$

That is, the Darboux Theorem provides a local model for a symplectic manifold near x ($\mathbb{R}^{2n}$) together with a normal form for the symplectic structure (formula (3)). A proof of the existence of Darboux charts can be found in many textbooks, as [16].

Other examples of normal forms would include Weinstein's splitting formula for Poisson manifolds, Poincare's linearization theorem for the flow of vector fields near hyperbolic equilibrium points, etc...

What is the MGS form. The Marle-Guillemin-Sternberg form (MGS) is a local model for a symplectic manifold supporting a Hamiltonian action of a Lie group near a group orbit, together with normal forms for the symplectic structure, the group action and the momentum map. It has been obtained in [12, 13, 3]. This is a key result that has been profitably used in a number of situations. In the more geometric side, it has allowed to proof several important results in the theory of singular symplectic reduction [32, 3, 28] and topological properties of the momentum map [21], and in the framework of Hamiltonian dynamics it has been used to obtain several results on the qualitative behavior of symmetric Hamiltonian systems (see last section). In this notes we present an introduction to the basic features of the MGS form together with its use in some of these applications. We will present the main ideas of the proofs, but the most technical details will be left aside for reasons of time, and we will refer to the relevant literature. The objective of these notes is to give the reader unfamiliar with these tools the necessary knowledge to understand the research literature on the subject.

1 Hamiltonian actions and the Witt-Artin decomposition

Hamiltonian actions. From now on (M, ω) will denote a smooth symplectic manifold. A vector field $X \in \mathfrak{X}(M)$ is called Hamiltonian if there is a function

$f \in C^\infty(M)$ (called the Hamiltonian) satisfying

$$\iota_X \omega = df.$$

There are many Hamiltonian functions for a given vector field, and they all differ by locally constant functions.

A symplectic vector space (V, Ω) is an even dimensional vector space with a non-degenerate antisymmetric bilinear form Ω. A symplectic vector space is naturally also a symplectic manifold with a constant symplectic form. Note that for every $x \in M$, T_xM is a symplectic vector space with $\Omega = \omega(x)$.

A particularly interesting situation arises when a Lie group acts on a symplectic manifold preserving the geometric data, that is, the symplectic form, in a particular way. In this notes we will assume that all actions are proper, unless otherwise stated.

Definition 1.1. *A (left) action $G \times M \to M$ of a Lie group G on the symplectic manifold (M, ω) is called Hamiltonian if*

- *(i) For all $g \in G$, $g^*\omega = \omega$,*
- *(ii) There is a map $J : M \to \mathfrak{g}^*$, called the momentum map of the action, satisfying*
$$\iota_{\xi_M}\omega = d\langle J, \xi\rangle, \quad \forall \xi \in \mathfrak{g}$$
- *(iii) For all $g \in G$ and $x \in M$, $J(g \cdot x) = \mathrm{Ad}^*_{g^{-1}} J(x)$,*

where $\mathfrak{g}$ is the Lie algebra of G, $\mathfrak{g}^*$ is its dual, $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is the canonical pairing, and $\xi_M \in \mathfrak{X}(M)$ is the fundamental vector field associated to $\xi \in \mathfrak{g}$, and defined by

$$\xi_M(x) = \frac{d}{dt}\Big|_{t=0} e^{t\xi} \cdot x.$$

Condition (i) means that G acts by symplectormophisms, and condition (ii) means that every fundamental vector field is a Hamiltonian vector field. Indeed, the Hamiltonian for ξ_M is the function J^ξ, defined by $J^\xi(x) = \langle J(x), \xi\rangle$ for any $x \in M$. Condition (iii) means that the momentum map is G-equivariant, with respect to the given action on M and the coadjoint representation on $\mathfrak{g}^*$. It is possible to show that this third condition is equivalent with J being a Poisson map between (M, ω) and $\mathfrak{g}^*$ equipped with the $(-)$ Lie-Poisson structure.

A Hamiltonian G-space is a cuadruple (M, ω, G, J), where (M, ω) is a symplectic manifold and G is a Lie group acting in a Hamiltonian fashion on M with associated equivariant momentum map $J : M \to \mathfrak{g}^*$

A particular case is given by the following situation. Let (V, Ω) be a symplectic vector space and let H be a compact Lie group acting linearly on V by symplectomorphisms. then this action is automatically Hamiltonian and has momentum map given by

$$\langle J_V(v), \xi\rangle = \frac{1}{2}\Omega(\xi \cdot v, v), \quad \forall v \in V,\ \xi \in \mathfrak{h} \tag{4}$$

where $\xi \cdot v = \frac{d}{dt}\big|_0 e^{t\xi} \cdot v$.

Note that if (M, ω, G, J) is a Hamiltonian G-space, then for any $x \in M$, the pair $(T_x M, \omega(x))$ is automatically a symplectic vector space on where the compact Lie group G_x acts linearly and in a Hamiltonian fashion.

The Witt-Artin decomposition. This is a standard construction in symplectic linear algebra that will be needed in the construction of the MGS form, as well as in its applications.

Let (M, ω, G, J) be a Hamiltonian G-space, and let $x \in M$ with $J(x) = \mu$. Denote by G_μ the stabilizer of μ with respect to the coadjoint representation of G on $\mathfrak{g}^*$ and by $\mathfrak{g}_\mu$ its Lie algebra. A Witt-Artin decomposition at x is a G_x-invariant splitting of $T_x M$ as

$$T_x M = T_1 \oplus T_2 \oplus W \oplus N, \tag{5}$$

where, denoting the symplectic perpendicular by ${}^\omega$:

(i) $T_1 = \mathfrak{g} \cdot x \cap (\mathfrak{g} \cdot x)^\omega$,

(ii) T_2 is a symplectic vector subspace of $(T_x M, \omega(x))$ and

$$T_1 \oplus T_2 = \mathfrak{g} \cdot x,$$

(iii) N is a symplectic vector subspace of $(T_x M, \omega(x))$ and

$$(\mathfrak{g} \cdot x)^\omega = T_1 \oplus N,$$

(iv) W is a Lagrangian complement to T_1 in $(N \oplus T_2)^\omega$.

Witt-Artin decompositions always exist for any $x \in M$. In fact they can be generalized to the situation where instead of $g \cdot x$ we start with an arbitrary isotropic G_x-invariant subspace. The following proposition gives some of the most important properties of the Witt-Artin decomposition. For a proof see Theorem 7.1.1 in [28].

Proposition 1.1. *Consider a Witt-Artin decomposition at* x *as in* (5)*. Then*

(i) The subspace $S := W \oplus N$ *is a slice at* x *as in the Koszul-Palais Tube Theorem.*

(ii) $T_1 = \mathfrak{g}_\mu \cdot x$.

(iii) $(\mathfrak{g} \cdot x)^\omega = \ker T_x J$*, and therefore* N *can be chosen to be any* G_x*-invariant complement to* $\mathfrak{g}_\mu \cdot x$ *in* $\ker T_x J$.

(iv) Recall that, since $J : M \to \mathfrak{g}^*$ *is* G*-equivariant, then* $G_x \subset G_\mu$*. Therefore, under the adjoint representation of* G*, the spaces* $\mathfrak{g}_x$ *and* $\mathfrak{g}_\mu$ *are* G_x*-invariant. Since* G_x *is compact, we can define a* G_x*-invariant splitting*

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}_x \oplus \mathfrak{q},$$

where $\mathfrak{g}_\mu = \mathfrak{m} \oplus \mathfrak{g}_x$. This induces the dual splitting of $\mathfrak{g}^*$, invariant under the restriction to G_x of the coadjoint representation of G,

$$\mathfrak{g}^* = \mathfrak{m}^* \oplus \mathfrak{g}_x^* \oplus \mathfrak{q}^*,$$

where $\mathfrak{g}_\mu^* = \mathfrak{m}^* \oplus \mathfrak{g}_x^*$ and we have use the following identifications to realize the dual spaces as subspaces of $\mathfrak{g}^*$:

$$\mathfrak{g}_\mu^* = \mathfrak{q}^\circ, \quad \mathfrak{m}^* = (\mathfrak{g}_x \oplus \mathfrak{q})^\circ, \quad \mathfrak{g}_x^* = (\mathfrak{m} \oplus \mathfrak{q})^\circ, \quad \mathfrak{q}^* = \mathfrak{g}_\mu^\circ.$$

In particular, $\mathfrak{m}^*$ is G_x-invariant. Then, the map $f : W \to \mathfrak{m}^*$ defined by

$$\langle f(w), \xi \rangle = \omega(x)(\xi_M(x), w), \tag{6}$$

for all $w \in W$ and $\xi \in \mathfrak{m}$, is a G_x-equivariant linear isomorphism.

Since by construction, N is a symplectic vector subspace of T_xM, call its symplectic form $\omega_N := \omega(x)\big|_N$. Recall also that since the G-action on M is Hamiltonian, then G_x acts on (N, ω_N) in a Hamiltonian fashion with an associated quadratic homogenous momentum map $\mathbf{J}_N$ given by the formula (4). The pair (N, ω_N) is called the symplectic normal space at x and it is the most important object in the Witt-Artin decomposition, since the Hamiltonian action of G on M is locally characterized by the linear symplectic representation of G_x on (N, ω_N).

2 The MGS form

As stated in the introduction, the MGS form is a local model for a Hamiltonian G-space $(M, \omega, G, \mathbf{J})$ that simultaneously puts into normal form the symplectic structure, group action and momentum map. To construct it, we will follow closely the approach of [28].

Let $x \in M$ with $\mathbf{J}(x) = \mu$ and consider the direct product

$$G \times (\mathfrak{m}^* \times N),$$

where N is the symplectic normal space at x corresponding to a Witt-Artin decomposition. Let $G_x \times (G \times (\mathfrak{m}^* \times N)) \to G \times (\mathfrak{m}^* \times N)$ be the action defined by

$$h \cdot (g, \rho, v) = (gh^{-1}, h \cdot \rho, h \cdot v). \tag{7}$$

This action is free, and since G_x is compact, the quotient space is a manifold, that we denote by

$$Y = G \times_{G_x} (\mathfrak{m}^* \times N),$$

and its elements are the equivalence classes $[g, \rho, v]$, where two elements in $G \times (\mathfrak{m}^* \times N)$ are declared equivalent if they belong to the same G_x-orbit.

There is also an action of G on $G \times (\mathfrak{m}^* \times N)$, given by

$$g' \cdot (g, \rho, v) = (g'g, \rho, v). \tag{8}$$

The actions of G and G_x clearly commute, therefore there is an induced G-action on the quotient Y given by

$$g' \cdot [g, \rho, v] = [g'g, \rho, v]. \tag{9}$$

We can endow Y with a differential two-form as follows. Denote by π the projection map

$$\begin{array}{rcl} \pi : G \times (\mathfrak{m}^* \times N) & \longrightarrow & G \times_{G_x} (\mathfrak{m}^* \times N) \\ (g, \rho, v) & \mapsto & [g, \rho, v]. \end{array}$$

Let us also use the left trivialization of TG, given by

$$\begin{array}{rcl} G \times \mathfrak{g} & \longrightarrow & TG \\ (g, \xi) & \mapsto & g\xi := T_e L_g(\xi) \in T_g G, \end{array}$$

where $L_g : G \times G \to G$; $L_g(g') = gg'$ is the left translation. Then, we can characterize the tangent space to $[g, \rho, v] \in Y$ as

$$T_{[g,\rho,v]}Y = \{\gamma_{\xi,\dot\rho,\dot v} := T_{(g,\rho,v)}\pi(g \cdot \xi, \dot\rho, \dot v) \, : \, \xi \in \mathfrak{g}, \, \dot\rho \in \mathfrak{m}^*, \, \dot v \in N\}. \tag{10}$$

Next, define the two-form ω_Y by

$$\begin{array}{rcl} \omega_Y([g,\rho,v])(\gamma_{\xi_1,\dot\rho_1,\dot v_1}, \gamma_{\xi_2,\dot\rho_2,\dot v_2}) & = & \langle \dot\rho_2 + T_v J_N(\dot v_2), \xi_1\rangle - \langle \dot\rho_1 + T_v J_N(\dot v_1), \xi_2\rangle \\ & & + \langle \rho + \mathbf{J}(v), [\xi_1, \xi_2]\rangle + \omega(x)(\xi_{1M}(x), \xi_{2M}(x)) \\ & & + \omega_N(\dot v_1, \dot v_2) \end{array} \tag{11}$$

We are now in a position of stating the main result of this section.

Theorem 2.1 (MGS form)**.** *Let (M, ω, G, J) be a Hamiltonian G-space. Let $x \in M$ with $J(x) = \mu$. Choose G_x-invariant splittings $\mathfrak{g}_\mu = \mathfrak{m} \oplus \mathfrak{g}_x$ and $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}_x \oplus \mathfrak{q}$ with dual splittings $\mathfrak{g}_\mu^* = \mathfrak{m}^* \oplus \mathfrak{g}_x^*$ and $\mathfrak{g}^* = \mathfrak{m}^* \oplus \mathfrak{g}_x^* \oplus \mathfrak{q}^*$. Let N be a G_x-invariant complement to $\mathfrak{g}_\mu \cdot x$ in $\ker T_x J$. Then, the two-form ω_Y, defined by* (11) *on*

$$Y = G \times_{G_x} (\mathfrak{m}^* \times N)$$

is a symplectic form near $[e, 0, 0]$. Furthermore: the G-action on Y defined by (9) *is Hamiltonian with momentum map $J_Y : Y \to \mathfrak{g}^*$ given by*

$$J_Y([g, \rho, v]) = \mathrm{Ad}^*_{g^{-1}}(\mu + \rho + J_N(v)). \tag{12}$$

There is a local symplectomorphism $\phi : Y \to M$ satisfying

- *(i) ϕ is equivariant with respect to the action* (9) *on Y and the original action on M.*
- *(ii) $\phi([e, 0, 0]) = x$.*

(iii) $\phi(G \times_{G_x} (\{0\} \times \{0\})) = G \cdot x$.

(iv) ϕ *is a symplectomorphism, i.e.* $\phi^*\omega = \omega_Y$.

Note: By local diffeomorphism we mean that there are G_x-invariant neighborhoods $0 \in U_1$ in $\mathfrak{m}^*$ and $0 \in U_2$ in N and a G-invariant neighborhood $G \cdot x \subset U'$ in M such that

$$\phi : G \times_{G_x} (U_1 \times U_2) \longrightarrow U'$$

is a diffeomorphim. That ω_Y is a symplectic form near $[e, 0, 0]$ means that U_1 and U_2 can be chosen small enough so that ω_Y is symplectic on $G \times_{G_x} (U_1 \times U_2)$.

Since ϕ is symplectic and G-equivariant, we have as a consequence

$$J_Y = J \circ \phi.$$

Therefore, Theorem 2.1 gives a local model for the Hamiltonian G-space (M, ω, G, J) near the orbit $G \cdot x$, (the local model being the space $Y = G \times_{G_x} (\mathfrak{m}^* \times N)$), a normal form for the group action (9), a normal form for the symplectic structure (11) and a normal form for the momentum map (12).

Theorem 2.1 can be used to study problems involving local questions about the geometry of Hamiltonian G-spaces, since we can replace the data (M, ω, G, J) near any point by the model Hamiltonian G-space given by the data (Y, ω_Y, G, J_Y).

Relative Darboux. Before proving Theorem 2.1 we will need to generalize the Darboux Theorem to the situation where there is a group action on M. See [28] for a proof.

Theorem 2.2 (Relative Darboux)**.** *Let M be a manifold and ω, ω' two symplectic forms on it. Let G be a Lie group acting on M by symplectormophisms with respect to both ω and ω'. Let $x \in M$ and suppose that for every $z \in G \cdot x$, $\omega(z) = \omega'(z)$. Then, there are two open G-invariant neighborhoods U, U' of $G \cdot x$ and a G-equivariant symplectomorphism $\phi : (U', \omega') \to (U, \omega)$ such that* $\phi\big|_{G \cdot x} = \mathrm{id}$.

Proof. (Of Theorem 2.1) We will give an outline of the proof. The details can be filled by looking at chapter 7 of [28].

First, fix a G_x-invariant complement N to $\mathfrak{g}_\mu \cdot x$ in $\ker T_x J$, and complete a Witt-Artin decomposition at x,

$$T_x M = T_1 \oplus T_2 \oplus W \oplus N.$$

Let $f : W \to \mathfrak{m}^*$ be the map introduced in (6). Then, define the map

$$\begin{array}{rcl} \phi_1 : G \times_{G_x} (\mathfrak{m}^* \times N) & \longrightarrow & G \times_{G_x} (W \oplus N) \\ {[g, \rho, v]} & \mapsto & [g, f^{-1}(\rho) + v]. \end{array}$$

Since f is invertible, this is a G-equivariant diffeomorphism. Recall from Proposition 1.1 (i) that $W \oplus N$ is a slice at x. Then, according to the Koszul-Palais Tube Theorem, there is local diffeomorphism

$$\phi_2 : G \times_{G_x} (W \oplus N) \longrightarrow M$$

which sends $[e, 0]$ to x, $G \times_{G_x} \{0\}$ to $G \cdot x$ and that it is equivariant.

Therefore, the composition $\phi_3 = \phi_2 \circ \phi_1 : G \times_{G_x} (\mathfrak{m}^* \times N) \to M$ is a local diffeomorphism which sends $[e, 0, 0]$ to x, $G \times_{G_x} (\{0\} \times \{0\})$ to $G \cdot x$ and that it is equivariant onto a G-invariant neighborhood U' of $G \cdot x$ in M. Let w' the symplectic form on U' defined by

$$\omega' = \phi_3^{-1^*} \omega_Y.$$

It is clear that G acts on U' by symplectomorphisms to both ω and ω'. It is possible to see that for all $z \in G \cdot x$, $\omega'(z) = \omega(z)$. Therefore, according to the Relative Darboux Theorem 2.2, near $G \cdot x$ there are two open G-invariant neighborhoods U'' and U''' of $G \cdot x$ in U' and a G-equivariant symplectomorphism $\phi_4 : (U'', \omega') \to (U''', \omega)$. It is straightforward to check that the local diffeomorphism

$$\phi : G \times_{G_x} (\mathfrak{m}^* \times N) \longrightarrow M$$

defined by $\phi = \phi_4 \circ \phi_3$ satisfies all the properties of the statement of Theorem 2.1. $\square$

3 Application 1. Singular symplectic reduction

We will start showing how to use the MGS form by an application to singular reduction. Recall that if the Lie group action in a Hamiltonian G-space (M, ω, G, J) is free, then $\mathbf{J}^{-1}(0)$ is a smooth submanifold of M, invariant by G. Let $\iota : J^{-1}(0) \hookrightarrow M$ denote the canonical inclusion and $\pi : J^{-1}(0) \to J^{-1}(0)/G$ the group projection. Since the inherited G-action on $J^{-1}(0)$ is also proper and free, the orbit space $J^{-1}(0)/G$ is a smooth manifold in the quotient topology. The Marsden-Weinstein reduction procedure [15] states that $J^{-1}(0)/G$ is a symplectic reduced manifold in a natural way, where its symplectic form, ω_{red} is uniquely defined by

$$\pi^* \omega_{\text{red}} = \iota^* \omega.$$

If the G-action is not free, and 0 is a singular value of J, neither $J^{-1}(0)$ nor $J^{-1}(0)/G$ are smooth manifolds. The singular reduction theorem states, however, that both are partitioned by smooth manifolds (strata), and that the strata of $J^{-1}(0)/G$ are equipped with natural reduced symplectic forms. We will use the MGS form to prove the following version of singular symplectic reduction.

Theorem 3.1. *Let (M, ω, G, J) be a Hamiltonian G-space. Suppose that the G-action is not free and that 0 is a singular value of J. Define, for each $H \subset G$ the (H)-orbit type set*

$$M_{(H)} = \{x \in M \, : \, G_x \quad \textit{is conjugate to} \quad H\}.$$

Then,

(i) The sets $J^{-1}(0) \cap M_{(H)}$, if not empty, are disjoint smooth G-invariant submanifolds of M, and their union over $H \subset G$ is a partition of $J^{-1}(0)$.

(ii) *The sets* $(J^{-1}(0)\cap M_{(H)})/G$, *if not empty, are disjoint smooth manifolds, and their union over* $H\subset G$ *is a partition of* $J^{-1}(0)/G$.

(iii) *For each* H *such that* $J^{-1}(0)\cap M_{(H)}$ *is not empty, denote by* $\iota_H : J^{-1}(0)\cap M_{(H)} \hookrightarrow M$ *and* $\pi_H : J^{-1}(0)\cap M_{(H)} \to (J^{-1}(0)\cap M_{(H)})/G$ *the smooth inclusion and projection, respectively. Then,* $(J^{-1}(0)\cap M_{(H)})/G$ *is a smooth symplectic reduced manifold with symplectic form* ω^H_{red} *defined by*

$$\pi_H^*\omega^H_{red} = \iota_H^*\omega.$$

Proof. Note that by definition, the sets $M_{(H)}$ are disjoint and their union is precisely M, so the only non-trivial statment of (i) is that a fixed non-empty $J^{-1}(0)\cap M_{(H)}$ is a G-invariant smooth submanifold of M. The strategy to prove this, as well as the rest of statements of the theorem, is to substitute (M,ω,G,J) by the model space (Y,ω_Y,G,J_Y) given by the MGS form. This is justified by the fact that all the properties that appear in the theorem are local.

To prove (i), let $x\in M$ with $G_x = H$ and $J(x) = 0$. Such a x exists, otherwise $J^{-1}(0)\cap M_{(H)}$ would be empty. Note that since $\mu = 0$, $G_\mu = G$ and then $\mathfrak{q} = 0$ and $\mathfrak{g} = \mathfrak{m}\oplus\mathfrak{g}_x$ in the splitting of $\mathfrak{g}$ used in the construction of the MGS form. Let $Y = G\times_H(\mathfrak{m}^*\times N)$ be the MGS local model space for which $[e,0,0]$ corresponds to x.

We claim that for $[g,\rho,v]\in Y$, $G_{[g,\rho,v]} = gH_{(\rho,v)}g^{-1}$, where $H_{(\rho,v)}$ denotes the stabilizer of $(\rho,v)\in\mathfrak{m}^*\times N$ according to the linear diagonal action of $H = G_x$. To see this, let $g'\in G$ and $[g,\rho,v]\in Y$. Then

$$\begin{aligned}
g'\in G_{[g,\rho,v]} \quad &\Leftrightarrow \quad g'\cdot[g,\rho,v] = [g,\rho,v]\\
&\Leftrightarrow \quad [g'g,\rho,v] = [g,\rho,v]\\
&\Leftrightarrow \quad \exists h\in H \quad \text{s.t} \quad (g'g,\rho,v) = (gh, h^{-1}\cdot\rho, h^{-1}\cdot v)\\
&\Leftrightarrow \quad g' = ghg^{-1} \quad \text{with} \quad h\in H_\rho\cap H_v = H_{(\rho,v)}
\end{aligned}$$

Therefore, $Y_{(H)} = G\times_H(\mathfrak{m}^{*H}\times N^H)$, where the symbol ^{H} denotes the set of fixed points. For instance,

$$N^H = \{v\in N \,:\, h\cdot v = v \quad \text{for all} \quad h\in H\},$$

and accordingly for $\mathfrak{m}^{*H}$. The fact that fixed points sets for linear actions are vector subspaces will be of fundamental importance in the following. To see that $Y_{(H)}$ is of this form, just compute

$$\begin{aligned}
Y_{(H)} &= \{[g,\rho,v]\in Y \,:\, G_{[g,\rho,v]} \quad \text{is conjugate to} \quad H\} \\
&= \{[g,\rho,v]\in Y \,:\, gH_{(\rho,v)}g^{-1} \quad \text{is conjugate to} \quad H\} \\
&= \{[g,\rho,v]\in Y \,:\, H_{(\rho,v)} \quad \text{is conjugate to} \quad H\} \\
&= \{[g,\rho,v]\in Y \,:\, H_{(\rho,v)} \quad \text{is equal to} \quad H\} \quad (\text{since} \quad H_{(\rho,v)}\subseteq H) \\
&= \{[g,\rho,v]\in Y \,:\, H_\rho\cap H_v \quad \text{is equal to} \quad H\} \\
&= \{[g,\rho,v]\in Y \,:\, H_\rho = H_v = H\} \\
&= \{[g,\rho,v]\in Y \,:\, h\cdot\rho=\rho,\, h\cdot v = v \quad \text{for all} \quad h\in H\} \\
&= \{[g,\rho,v]\in Y \,:\, \rho\in\mathfrak{m}^{*H},\, v\in N^H\} \\
&= G\times_H(\mathfrak{m}^{*H}\times N^H)
\end{aligned}$$

The next step is to compute the zero level set of J_Y. For that, recall from the normal form of the momentum map (12) that

$$\begin{aligned}
J_Y^{-1}(0) &= \{[g,\rho,v]\in Y \,:\, \mathrm{Ad}^*_{g^{-1}}(\rho+J_N(v))=0\} \\
&= \{[g,\rho,v]\in Y \,:\, \rho+J_N(v)=0\} \\
&= \{[g,\rho,v]\in Y \,:\, \rho=0,\, J_N(v)=0\},
\end{aligned}$$

where the last equality follows from the facts that the coadjoint representation is linear, $\rho\in\mathfrak{m}^*$, $J_N(v)\in\mathfrak{h}^*$ and the sum $\mathfrak{m}^*\oplus\mathfrak{h}^*$ is direct (and equal to $\mathfrak{g}^*$, since recall that $\mathfrak{h}=\mathfrak{g}_x$ and $\mathfrak{g}_\mu=\mathfrak{g}$).

To get (i), we want to prove that $\mathbf{J}_Y^{-1}(0)\cap Y_{(H)}$ is a submanifold of Y. Recall that the quadratic momentum map J_N is defined by

$$\langle J_N(v),\xi\rangle = \frac{1}{2}\omega_N(\xi\cdot v, v), \quad \xi\in\mathfrak{h},\, v\in N.$$

Therefore, for every $v\in N^H$, $J_N(v)=0$ (since $\xi\cdot v=0$). This implies that

$$\begin{aligned}
J_Y^{-1}(0)\cap Y_{(H)} &= \{[g,\rho,v] \,:\, \rho=0,\, v\in N^H\} \\
&= G\times_H(\{0\}\times N^H).
\end{aligned}$$

Notice that since N^H is a vector subspace of N, $J_Y^{-1}(0)\cap Y_{(H)}$ is a smooth embedded submanifold of $Y=G\times_H(\mathfrak{m}^*\times N)$. It is also obviously G-invariant under the action (9).

The only non-trivial thing in (ii) is to show that $(J_Y^{-1}(0)\cap Y_{(H)})/G$ is a smooth manifold. To see this, recall that the H and G-actions defined by (7) and (8) commute. Therefore

$$\begin{aligned}
(J_Y^{-1}(0)\cap Y_{(H)})/G &= (G\times_H(\{0\}\times N^H))/G = (G\times_G(\{0\}\times N^H))/H \\
&= (\{0\}\times N^H)/H = N^H/H = N^H.
\end{aligned}$$

Since N^H is smooth, $(J_Y^{-1}(0)\cap Y_{(H)})/G$ is smooth and (ii) follows.

To prove (iii), we claim that the reduced symplectic structure on $(J_Y^{-1}(0)\cap Y_{(H)})/G = N^H$ is given precisely by $\omega^H_{\text{red}} := \omega_N\big|_{N^H}$.

To see this, note that the canonical inclusion $\iota_H : J_Y^{-1}(0)\cap Y_{(H)} \hookrightarrow Y$ is given by the identity on Y, while the projection

$$\pi_H : J_Y^{-1}(0)\cap Y_{(H)} = G\times_H (\{0\}\times N^H) \to (J_Y^{-1}(0)\cap Y_{(H)}))/G = N^H$$

is given by

$$\pi_H([g,0,v]) = v.$$

Recall from (10) that the tangent spaces to $(J_Y^{-1}(0)\cap Y_{(H)})/G$ are given by

$$T_{[g,0,v]}(J_Y^{-1}(0)\cap Y_{(H)}) = \{\gamma_{\xi,\dot\rho,\dot v}\in T_{[g,0,v]}Y \;:\; \dot\rho = 0,\, \dot v\in N^H\}.$$

Let $\gamma_{\xi,0,\dot v}\in T_{[g,0,v]}(J_Y^{-1}(0)\cap Y_{(H)})$. Then

$$T_{[g,0,v]}\pi_H(\gamma_{\xi,0,\dot v}) = v\in T_vN^H = N^H.$$

We can now prove that $\omega^H_{\text{red}} = \omega_N\big|_{N^H}$ is the solution of $\pi_H^*\omega^H_{\text{red}} = \iota_H^*\omega_Y$. For that, let $\gamma_{\xi_1,0,\dot v_1},\gamma_{\xi_2,0,\dot v_2}\in T_{[g,0,v]}(J_Y^{-1}(0)\cap Y_{(H)})$. Computing the left hand side we obtain

$$(\pi_H^*\omega_N\big|_{N^H})(\gamma_{\xi_1,0,\dot v_1},\gamma_{\xi_2,0,\dot v_2}) = \omega_N(\dot v_1,\dot v_2). \tag{13}$$

If we compute the right hand side we get, using (11),

$$\begin{aligned}(\iota_H^*\omega_Y)(\gamma_{\xi_1,0,\dot v_1},\gamma_{\xi_2,0,\dot v_2}) &= \langle T_vJ_N(\dot v_2),\xi_1\rangle - \langle T_vJ_N(\dot v_1),\xi_2\rangle\\ &\quad +\omega(x)(\xi_{1M}(x),\xi_{2M}(x)) + \omega_N(\dot v_1,\dot v_2)\\ &= \omega_N(\dot v_1,\dot v_2) \qquad (14)\end{aligned}$$

The last equality follows from the following facts: First, from (4) it follows that

$$\langle T_vJ_N(\dot v),\eta\rangle = \omega_N(\eta\cdot\dot v, v),\quad v,\dot v\in N,\ \eta\in\mathfrak{h}.$$

Therefore, $T_vJ_N(\dot v) = 0$ if $\dot v\in N^H$ (this is a particular case of the Bifurcation Lemma, [2]). Also, $\omega(x)(\xi_{1M}(x),\xi_{2M}(x)) = 0$, since

$$\begin{aligned}\omega(x)(\xi_{1M}(x),\xi_{2M}(x)) &= d\langle J(\cdot),\xi_1\rangle(\xi_{2M})(x)\\ &= \frac{d}{dt}\Big|_{t\,=\,0}\langle J(\phi_{e^{t\xi_2}}(x)),\xi_1\rangle\\ &= \frac{d}{dt}\Big|_{t\,=\,0}\langle \mathrm{Ad}^*_{e^{-t\xi_2}}J(x),\xi_1\rangle\\ &= \frac{d}{dt}\Big|_{t\,=\,0}\langle \mathrm{Ad}^*_{e^{-t\xi_2}}0,\xi_1\rangle\\ &= 0.\end{aligned}$$

Then, (13) together with (14) proof that $\omega^H_{\text{red}} = \omega_N\big|_{N^H}$ is the solution of $\pi_H^*\omega^H_{\text{red}} = \iota_H^*\omega_Y$. To see that this form is actually symplectic, we use the fact that the fixed-points set of a symplectic vector field by linear symplectic representation of a compact Lie group is a symplectic vector subspace. This is a particular case of a result of Guillemin and Sternberg [11]. Therefore $\omega^H_{\text{red}} = \omega_N\big|_{N^H}$ is non-degenerate (and constant), hence symplectic, proving (iii). □

Note: The MGS form can also be used to prove very sophisticated topological features of the singular quotient $J^{-1}(0)/G$. Namely that its stratification alluded in Theorem 3.1 admits a smooth singular atlas, satisfies the Whitney conditions, and has a cone structure. These are more advanced topics that lie out of the scope of these notes, and can be found in [32, 3, 28].

4 Application 2. Persistence of relative equilibria

In the second application of the MGS form we will see how it can be profitably used in the qualitative study of the dynamics of symmetric Hamiltonian systems. We will prove a classic result which goes back to Arnold that says that at a relative equilibrium of a symmetric Hamiltonian system with G_μ compact and Abelian, there exist relative equilibria with momenta any prescribed value close to μ. This is a typical persistence result, since it requires as hypothesis a certain non-degeneracy assumption.

Hamiltonian systems and relative equilibria. Recall that if (M, ω) is a symplectic manifold and $h \in C^\infty(M)$ a function, there is an associated Hamiltonian vector field X_h uniquely defined by Hamilton's equations

$$\iota_{X_h}\omega = dh. \tag{15}$$

The Hamiltonian dynamics corresponding to h is then the flow of X_h, and we call (M, ω) the phase space of the dynamics.

Suppose furthermore that we have a Hamiltonian G-space and that h is a G-invariant function. Therefore the Hamiltonian vector field X_h defined by (15) is invariant (a.k.a. symmetric), and its flow is equivariant. The quintuple (M, ω, G, J, h) is called a G-Hamiltonian system.

We would like to understand the dynamics of this flow, which may be very complicated. The point of view adopted by the qualitative study of symmetric Hamiltonian systems is to identify a special family of solutions (integral curves) of this flow, and then to carry out a local analysis of the properties of the flow near that solution. For this kind of symmetric Hamiltonian systems, the special family of solutions under consideration is that of relative equilibria.

Definition 4.1. *Let (M, ω, G, J, h) be a G-Hamiltonian system and $x \in M$ a point such that $J(x) = \mu$ and the solution of X_h through x is contained in the group orbit $G \cdot x$. Then x is called a relative equilibrium.*

The reason for choosing relative equilibria of G-Hamiltonian systems as the starting point for a qualitative analysis is that it is believed that relative equilibria act as "organizing centers" for the dynamics of the Hamiltonian flow. That is, it suffices to study the flow at and near relative equilibria to understand to a big extent the properties of the flow. The next proposition collects some properties of relative equilibria. For a proof, see [1, 14].

Proposition 4.1. *In the setup of Definition 4.1, the following are equivalent.*

(i) $x \in M$ *is a relative equilibrium with momentum* $J(x) = \mu$.

(ii) Every point in the solution (integral curve of X_h*) through* x *is also a relative equilibrium.*

(iii) Every point in the group orbit $G \cdot x$ *is also a relative equilibrium.*

(iv) There is an element $\xi \in \mathfrak{g}_\mu$ *such that the solution through* x *is given by*

$$x(t) = e^{t\xi} \cdot x.$$

The element ξ *is called a velocity of the relative equilibrium* x*. If* x *admits* 0 *as a velocity, then it is a fixed equilibrium.*

(v) x *is a critical point of the augmented Hamiltonian*

$$h_\xi(x') = h(x') - \langle J(x'), \xi \rangle,$$

where ξ *is a velocity of the relative equilibrium* x.

(vi) The flow of X_h *drops to (continuous) flows on the symplectic reduced space* $J^{-1}(\mu)/G_\mu$ *and on the Poisson reduced space* M/G *for which the solution trough* x *is mapped onto fixed equilibria.*

The condition of regularity needed to state Arnold's persistence result is the following.

Definition 4.2. *In the previous setup let* $x \in M$ *be a relative equilibrium with momentum* $J(x) = \mu$ *and velocity* ξ*. Let* $N \subset T_x M$ *be some* G_x*-invariant complement to* $\mathfrak{g}_\mu \cdot x$ *in* $\ker T_x J$ *(a symplectic normal space). The relative equilibrium* x *is called non-degenerate if the bilinear form*

$$\mathrm{Hess}_x h_\xi \Big|_N$$

is non-degenerate.

Note that this property of non-degeneracy is well defined since by Proposition 4.1 (v), h_ξ has a critical point at x, so h_ξ has a well defined Hessian. If the restriction of this Hessian to N is non-degenerate for some choice of N then it will be non-degenerate for any other choice N'. To see this, let $v_1, v_2 \in N$ and $\xi_1, \xi_2 \in \mathfrak{g}_\mu$, and let $V_{1,2}$ be vector fields on M satisfying $V_{1,2}(0) = v_{1,2}$, with flows $F^t_{1,2}$. Then

$$\mathrm{Hess}_x h_\xi(v_1 + {\xi_1}_M(x), v_2 + {\xi_2}_M(x)) = \frac{d}{dt}\Big|_{t\,=\,0} \frac{d}{ds}\Big|_{s\,=\,0} \left(h_\xi\left(F_1^t(F_2^t(x))\right) + h_\xi\left(e^{t\xi_1} \cdot (e^{s\xi_2} \cdot x)\right)\right)$$

$$= \frac{d}{dt}\Big|_{t\,=\,0} \frac{d}{ds}\Big|_{s\,=\,0} \left(h(F_1^t(F_2^t(x)) - \langle J\left(F_1^t(F_2^t(x))\right) + \langle J\left(e^{t\xi_1} \cdot (e^{s\xi_2} \cdot x)\right), \xi \rangle\right)$$

since h is G-invariant

$$= \frac{d}{dt}\Big|_{t\,=\,0}\frac{d}{ds}\Big|_{s\,=\,0}\left(h(F_1^t(F_2^t(x))) - \langle J(F_1^t(F_2^t(x)), \xi\rangle - \langle \mathrm{Ad}^*_{e^{-t\xi_1}}\left(\mathrm{Ad}^*_{e^{-s\xi_e}}J(x)\right), \xi\rangle\right)$$

since J is equivariant

$$= \frac{d}{dt}\Big|_{t\,=\,0}\frac{d}{ds}\Big|_{s\,=\,0}\left(h(F_1^t(F_2^t(x)) - \langle J(F_1^t(F_2^t(x)), \xi\rangle - \langle J(x), \xi\rangle\right)$$

since $J(x) = \mu$ and $\xi_1, \xi_2 \in \mathfrak{g}_\mu$

$$= \mathrm{Hess}_x h_\xi(v_1, v_2).$$

The bundle equations. The strategy for studying relative equilibria via the MGS form is to pose the problem in the local model space Y. For that, suppose that we have a G-Hamiltonian system (M, ω, G, J, h) and a point $x \in M$ with $J(x) = \mu$. Assume that the action is free and that G_μ is compact, so that $G_x = e$ and then $\mathfrak{m}^* = \mathfrak{g}_\mu^*$ and $J_N = 0$. Then local model is given by

$$Y = G \times \mathfrak{g}_\mu^* \times N$$

and

$$J_Y(g, \rho, v) = \mathrm{Ad}^*_{g^{-1}}(\mu + \rho). \tag{16}$$

Using the local G-equivariant symplectomorphism

$$\phi : Y = G \times \mathfrak{g}_\mu^* \times N \longrightarrow M$$

we can pull-back to Y the hamiltonian, obtaining a function $h_Y := \phi^* h$. Note that since h is G-invariant, so is h_Y, and then according to the group action on Y (9) we have

$$h_Y(g, \rho, v) = h_Y(\rho, v), \quad \forall g \in G,\ \rho \in \mathfrak{g}_\mu^*,\ v \in N,$$

so h_Y is actually a function on $\mathfrak{g}_\mu^* \times N$.

The bundle equations are just Hamilton's equations for the model G-Hamiltonian system $(Y, \omega_Y, G, J_Y, h_Y)$. They have been obtained in [29, 30, 28], and in the case under study, where the action is free, they are given by

$$\begin{aligned} \dot{g} &= g D_{\mathfrak{g}_\mu^*} h_Y(\rho, v) && (17)\\ \dot{\rho} &= \mathrm{ad}^*_{D_{\mathfrak{g}_\mu^*} h_Y(\rho, v)} \rho && (18)\\ \dot{v} &= \omega_N^\sharp D_N h_Y(\rho, v) && (19) \end{aligned}$$

Persistence of relative equilibria. The persistence result that we want to proof is given by the following theorem. It was observed originally by Arnold, and then generalized by Patrick [23]. More recently it has been extended to the case when the Hamiltonian action of G is not free in [7, 27].

Theorem 4.1. *Let* (M, ω, G, J, h) *a* G*-Hamiltonian system. Let* $x \in M$ *be a non-degenerate relative equilibrium with momentum* μ *and velocity* ξ*. Suppose that* G *acts freely on* M *and that* G_μ *is a torus. Then, for every* $\mu' \in \mathfrak{g}^*$ *near* μ*, the level set* $\mathbf{J}^{-1}(\mu')$ *contains a relative equilibrium.*

Proof. We will substitute our G-Hamiltonian system with the corresponding model system near x, given by $(Y, \omega_Y, G, J_Y, h_Y)$. We start by identifying the conditions for $(g, \rho, v) \in Y$ to be a relative equilibrium. Recall from Proposition 4.1 (iv) and the expression for group action on Y (9) that (g, ρ, v) is a relative equilibrium if and only if there exists some $\xi' \in \mathfrak{g}_\mu$ such that

$$(g(t), \rho(t), v(t)) = (e^{t\xi'} g, \rho, v)$$

which is equivalent to

$$(\dot{g}, \dot{\rho}, \dot{v}) = (\xi' g, 0, 0).$$

Using the bundle equations (17), (18) and (19) we obtain

$$\begin{aligned} gD_{\mathfrak{g}^*_\mu} h_Y(\rho, v) &= \xi' g \\ \mathrm{ad}^*_{D_{\mathfrak{g}^*_\mu} h_Y(\rho, v)} \rho &= 0 \\ \omega_N^\sharp D_N h_Y(\rho, v) &= 0. \end{aligned}$$

Since G_μ is Abelian, and $D_{\mathfrak{g}^*_\mu} h_Y(\rho, v) \in \mathfrak{g}_\mu$ and $\rho \in \mathfrak{g}^*_\mu$, the second condition is automatically satisfied. Since ω_N is non-degenerate, the third condition is equivalent to $D_N h_Y(\rho, v) = 0$. Therefore, a point $(g, \rho, v) \in Y$ is a relative equilibrium if and only if

$$D_N h_Y(\rho, v) = 0, \tag{20}$$

and then the velocity ξ' of this relative equilibrium is automatically determined from the first condition as

$$\xi' = gD_{\mathfrak{g}^*_\mu} h_Y(\rho, v) g^{-1} = \mathrm{Ad}^*_g D_{\mathfrak{g}^*_\mu} h_Y(\rho, v).$$

Recall that since $\phi : Y \to M$ maps $(e, 0, 0)$ to x, the hypotheses of Theorem 4.1 imply that

$$D_N h_Y(0, 0) = 0 \quad \text{and} \quad \xi = D_{\mathfrak{g}^*_\mu} h_Y(0, 0). \tag{21}$$

Let us examine the meaning of the non-degeneracy condition of x in the model space Y. It is possible to show, and it is left as an exercise that

(i) $T_{(e,0,0)}\phi$ maps isomorphically $T_{(e,0,0)}(\{e\} \times \{0\} \times N) = N$ to $N' \in T_x M$, where N' is some symplectic normal space at x, possibly different than N.

(ii) $\phi^* \mathrm{Hess}_x h_\xi\big|_{N'} = D^2_N h_Y(0, 0)$.

Since the non-degeneracy hypothesis for x does not depend on the choice of symplectic normal space (see the comment after Definition 4.2), the hypotheses of Theorem 4.1 imply that $D_N^2 h_Y(0,0)$ is non-degenerate. Note from the expression (16) for the momentum map that every momentum value μ' of J_Y near μ can be written as

$$\mu' = \mathrm{Ad}^*_{g^{-1}}(\mu + \rho)$$

for some (non-unique) choices of $g \in G$ and $\rho \in \mathfrak{g}_\mu^*$ near 0. Therefore, using the relative equilibrium conditions on Y (21), the theorem will be proved if we show that, for every $\rho \in \mathfrak{g}_\mu^*$ close to 0, there is a point (g, ρ, v) such that

$$D_N(\rho, v) = 0, \tag{22}$$

under the hypothesis that

$$D_N h_Y(0,0) = 0, \quad \text{and} \quad D_N^2 h_Y(0,0) \quad \text{is non-degenerate.} \tag{23}$$

But this is now a trivial task, since by the Implicit Function Theorem, (23) implies that there is a smooth locally injective map

$$f : \mathfrak{g}_\mu^* \longrightarrow N$$

satisfying

(i) $f(0) = 0$, and

(ii) for every $\rho \in \mathfrak{g}_\mu^*$ near 0, $(\rho, f(\rho))$ solves (22), i.e.

$$D_N h_Y(\rho, f(\rho)) = 0.$$

Therefore, Theorem (4.1) is proved in the following way: Let $\mu' \in \mathfrak{g}_\mu^*$ be close to μ. Then, it is possible to find $g \in G$ and $\rho \in \mathfrak{g}_\mu^*$ near 0 such that

$$\mu' = \mathrm{Ad}^*_{g^{-1}}(\mu + \rho),$$

and then the point $(g, \rho, f(\rho)) \in Y$ is a relative equilibrium in the level set $J_Y^{-1}(\mu')$. □

Note: In [23] it has been proved that the collection of all the relative equilibria predicted by Theorem 4.1 form a local symplectic submanifold of M. To prove this is an easy exercise using the model symplectic form (11) that we leave to the reader.

5 Further comments

These notes have presented only a basic introduction to the Marle-Guillemin-Sternberg form ad its applications. In this section we will outline some other directions and open problems so that the interested reader can go deeper into the subject.

Normal forms for other related geometries. An equivalent construction to the MGS form has been carried out in [8] for Hamiltonian actions in the context of contact geometry, however as far as we are aware, there are not local models for Hamiltonian actions in other geometries like Poisson, Dirac or generalized complex geometry. It would probably be interesting to find such models in view of the recent advances in the understanding of Hamiltonian actions and momentum maps [9, 10, 4].

MGS in presence of additional structures. There are many symplectic manifolds of interest that present additional structure, like Kahler manifolds (additional Riemannian and complex compatible structures) or cotangent bundles (additional fibered structure). If we have a Hamiltonian G-space with additional structure that is also preserved under the group action, it would be interesting to develop local models and normal forms that include these structures. For instance, in [31] it has been obtained a MGS form for cotangent bundles $M = T^*Q$ in which the space Y is a fiber bundle over a local model for Q and the map $\phi : Y \to T^*Q$ is a local bundle isomorphism, so all the structure present in the Hamiltonian G-space is also present in the MGS form. This model can be obtained only for points in $M = T^*Q$ satisfying $G_\mu = G$ for their momentum values. An interesting problem would be to extend this construction to arbitrary points. Some steps in this direction have been done in [26] at a linear level. In addition, these models could be used to study the topology of the coisotropic stratification of reduced cotangent bundles introduced in [25].

Constructiveness of the model. An important feature of the MGS form is that the map $\phi : Y \to M$ is not constructive, since its existence follows from the relative Darboux Theorem. We are not aware of any situation or particular example in which this map has been explicitly constructed, with the exception of the MGS form for cotangent bundles obtained in [31]. It would be interesting to see if in some particular cases, notably in the situation of additional structure, this map can be made explicit.

MGS and relative equilibria/ relative normal modes. The bundle equations, which govern G-Hamiltonian systems on the MGS model space have been stated in the more general case when G is not compact and does not act freely in [30] (see also [28]). Below the persistence result in Theorem 4.1 there are many other results in the literature about persistence, bifurcations and stability of relative equilibria and relative normal modes that use to some extent the convenience of the MGS form [5, 7, 18, 19, 24, 20, 21, 27]. It seems that the bundle equations can provide a good framework to organize and advance this theory of qualitative analysis for the dynamics of G-Hamiltonian vector fields and it would be expected that new results could be obtained in a future using this tool.

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