Dynamics of a nonhomogeneous rolling ball and its impulsive control

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Outline

1. Kinematics of the rolling ball
2. Dynamics of a symmetric rolling ball
3. Optimal control
4. Impulsive approximation to optimal control

Outline

Description of the model: kinematics

Consider a ball of unit radius rolling on a flat surface (plate, ground).

A configuration is an element $(x, A) \in \mathbb{R}^2 \times \mathrm{SO}(3)$ (position and orientation).

If the canonical basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is attached to the ball at its original position, then these vectors get rotated to $\{A\mathbf{e}_1, A\mathbf{e}_2, A\mathbf{e}_3\}$

Recall the usual isomorphism $\hat{}: \mathbb{R}^3 \to \mathfrak{so}(3)$ given by

$$v \mapsto \begin{bmatrix} 0 & -v^3 & v^2 \\ v^3 & 0 & -v^1 \\ -v^2 & v^1 & 0 \end{bmatrix}$$

The spatial angular velocity $\omega \in \mathbb{R}^3$ is defined through $\hat{\omega} = \dot{A}A^{-1}$.

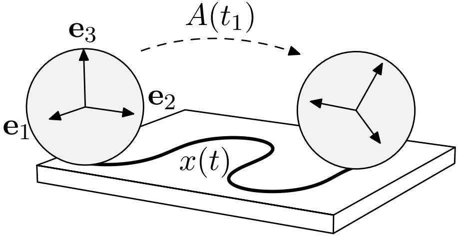
Nonholonomic constraints:

- No slipping: $\omega \times \mathbf{e}_3 = \dot{x}$ (velocity of the contact point on the ball is zero)
- No spinning: $\omega^3 = \langle \omega, \mathbf{e}_3 \rangle = 0$ (Veselova's constraint)

These can be combined into

- $\omega = \mathbf{e}_3 \times \dot{x}$

- One of the main consequences of including the “no spinning” condition is that if you make the ball roll along a given path $x(t)$ on the ground, then its orientation $A(t)$ is uniquely determined.



Principal bundle and principal connection

- Configuration space $\mathbb{R}^2 \times \mathrm{SO}(3)$
- Action of the group $\mathrm{SO}(3)$ on the configuration space:

$$g \cdot (x, A) = (x, Ag^{-1})$$

- Quotient space: $\mathbb{R}^2$
- The rolling constraint is $\mathrm{SO}(3)$-invariant and complementary to the vertical distribution

- Principal connection, with connection 1-form $\mathcal{A}\colon T(\mathbb{R}^2 \times \mathrm{SO}(3)) \to \mathfrak{so}(3) \equiv \mathbb{R}^3$ given by

$$\mathcal{A}(x, A, \dot{x}, \dot{A}) = A^{-1}(\widehat{\mathbf{e}_3 \times \dot{x}})A - A^{-1}\dot{A} =$$

$$\text{(body angular velocity consistent with } \dot{x}\text{)}$$
$$-$$
$$\text{(actual body angular velocity)}$$

- Allowed motions: $\mathcal{A} = 0$ ("horizontal", in principal bundle terms)
- Given a path $x(t)$, the resulting reorientation is computed by taking the horizontal lift of $x(t)$.

Curvature two-form

Let $X, Y \in T_x\mathbb{R}^2 \hookrightarrow T\mathbb{R}^3$. Consider their horizontal lifts

$$X^h_{(x,A)} = \left(x, A, X, (\widehat{\mathbf{e}_3 \times X})A\right)$$
$$Y^h_{(x,A)} = \left(x, A, Y, (\widehat{\mathbf{e}_3 \times Y})A\right).$$

The curvature two-form $B = d\mathcal{A} - [\mathcal{A}, \mathcal{A}]$ of the principal connection $\mathcal{A}$ for these two vectors turns out to be

$$B(x,A)\left(X^h_{(x,A)}, Y^h_{(x,A)}\right) = -A^{-1}\left(X \times Y\right) \in \mathbb{R}^3$$

Outline

Dynamics of a symmetric rolling ball

A material ball of radius r and mass M is called *symmetric* if its center of mass coincides with its geometric center and its principal moments of inertia $I_1,\ I_2,\ I_3$ satisfy $I_1 = I_2$.

Normalize the problem putting $I_1 = I_2 = 1$ and $r = 1$.
Define $z(t) = A(t)\mathbf{e}_3 \in \mathrm{S}^2$.

The spatial angular velocity can be written as $\omega = v_0 z + z \times \dot{z}$, so $v_0 = \langle \omega, z \rangle$ is its component along z.

The kinetic energy of the actual motion of the symmetric ball is

$$E = \frac{1}{2}(I_1 + Mr^2)\dot{z}^2 + \frac{1}{2}(I_3 + Mr^2)v_0^2,$$

and it is a preserved quantity.

Equations of Motion

For a general rigid body the momentum of the forces of inertia, $A(I\dot{\Omega} - I\Omega \times \Omega)$, where $\Omega = A^{-1}\omega$ is the body angular velocity, must be compensated by the torque due to the forces of the constraints. For the case of the torque induced by the nonholonomic constraints, we obtain the equation of motion

$$A(I\dot{\Omega} - I\Omega \times \Omega) \times e_3 = -M(\dot{\omega} \times e_3).$$

For a symmetric ball with $I_1 = I_2 = 1$, we have $I\Omega = (\Omega_1, \Omega_2, I_3\Omega_3) \equiv (\Omega_1, \Omega_2, I_3 v_0) = \Omega - (1 - I_3)v_0 e_3$, from which we obtain $I\dot{\Omega} = \dot{\Omega} - (1 - I_3)\dot{v}_0 e_3$. Then we have $A(I\dot{\Omega} - I\Omega \times \Omega) = \dot{\omega} - (1 - I_3)\dot{v}_0 z + (1 - I_3)v_0 z \times (v_0 z + z \times \dot{z})$, where we have used the equalities $\omega = A\Omega$, $\dot{\omega} = A\dot{\Omega}$, $\omega = v_0 z + z \times \dot{z}$.

Finally, we get $A(I\dot{\Omega} - I\Omega \times \Omega) = \dot{\omega} - (1 - I_3)(v_0 z)\dot{}$, and because of the equality $\dot{\omega} - (1 - I_3)(v_0 z)\dot{} = I_3\dot{\omega} + (1 - I_3)(z \times \ddot{z})$ the balance of momentum equation becomes

$$I_3(\dot{\omega} \times e_3) + (1 - I_3)(z \times \ddot{z}) \times e_3 = -M(\dot{\omega} \times e_3).$$

Then the system of dynamical plus constraint equations is

$$\begin{aligned} (I_3 + M)(\dot{\omega} \times e_3) + (1 - I_3)(z \times \ddot{z}) \times e_3 &= 0 \\ \omega &= v_0 z + z \times \dot{z} \\ \omega_3 &= 0. \end{aligned}$$

Define $u = \dot{z} \times z$. Following Cendra and Etchechoury [2006], it is possible to write this system in the variables (z, u, v_0) as

$$\dot{z}_1 = z_2 u_3 - z_3 u_2 \tag{1}$$

$$\dot{z}_2 = z_3 u_1 - z_1 u_3 \tag{2}$$

$$\dot{z}_3 = z_1 u_2 - z_2 u_1 \tag{3}$$

$$z_2 \dot{u}_1 - z_1 \dot{u}_2 = \lambda v_0 u_3 \tag{4}$$

$$0 = u_3 - v_0 z_3 \tag{5}$$

$$0 = u_1^2 + u_2^2 + u_3^2 + \lambda v_0^2 - \mu \tag{6}$$

$$0 = z_1^2 + z_2^2 + z_3^2 - 1 \tag{7}$$

$$0 = z_1 u_1 + z_2 u_2 + z_3 u_3, \tag{8}$$

where $\lambda = (I_3 + M)/(1 + M)$ and equation (6) represents conservation of energy, $\mu = 2E/(I_1 + Mr^2)$.

Remark

$\lambda = (I_3 + M)/(1 + M)$ measures how far the ball is from being homogeneous. A value of $\lambda = 1$ represents a homogeneous ball, and meaningful values are between these extreme cases:

- Equatorial disk: $I_1 = I_2 = 1$, $I_3 = 2$.

  $$\lambda = \frac{2 + M}{1 + M}$$

  It is the same λ as for the equatorial "ring".

- Polar bar: $I_1 = I_2 = 1$, $I_3 = 0$.

  $$\lambda = \frac{M}{1 + M}$$

Equations of Motion on $\mathrm{S}^2 \times \mathrm{S}^1$

For each given, positive energy value, equations (5)–(8) define a submanifold N of $\mathbb{R}^7$ diffeomorphic to $\mathrm{S}^2 \times \mathrm{S}^1$ that can be parameterized by the angles (φ, θ, ψ) as follows:

$$
\begin{aligned}
z_1 &= \sin\theta\cos\varphi \\
z_2 &= \sin\theta\sin\varphi \\
z_3 &= \cos\theta \\
u_1 &= -a\cos(\varphi-\psi)\cos^2\theta\cos\varphi - b\sin(\varphi-\psi)\sin\varphi \\
u_2 &= -a\cos(\varphi-\psi)\cos^2\theta\sin\varphi + b\sin(\varphi-\psi)\cos\varphi \\
u_3 &= a\cos(\varphi-\psi)\cos\theta\sin\theta \\
v_0 &= a\cos(\varphi-\psi)\sin\theta,
\end{aligned}
$$

where

$$a = \sqrt{\frac{\mu}{\lambda\sin^2\theta + \cos^2\theta}}, \quad b = \sqrt{\mu}.$$

The differential algebraic system (1)–(8) is equivalent to an analytic ODE on $\mathrm{S}^2 \times \mathrm{S}^1$:

$$\begin{aligned}
\dot{\theta} &= -b \sin(\varphi - \psi) \\
\dot{\varphi} &= -a \frac{\cos\theta}{\sin\theta} \cos(\varphi - \psi) \\
\dot{\psi} &= (b - a) \frac{\cos\theta}{\sin\theta} \cos(\varphi - \psi),
\end{aligned}$$

where φ and θ parameterize $z \in \mathrm{S}^2$, and $(\cos\psi, \sin\psi) \in \mathrm{S}^1$. Even though (φ, θ) are not, of course, global coordinates for S^2, the equations of motion define nevertheless an analytic vector field on the whole manifold $\mathrm{S}^2 \times \mathrm{S}^1$. A trajectory with $\theta = 0$ for the initial condition is necessarily a straight line. That is, the point z is initially on the top of the sphere and then it describes a circular motion on a vertical plane containing the center of the ball.

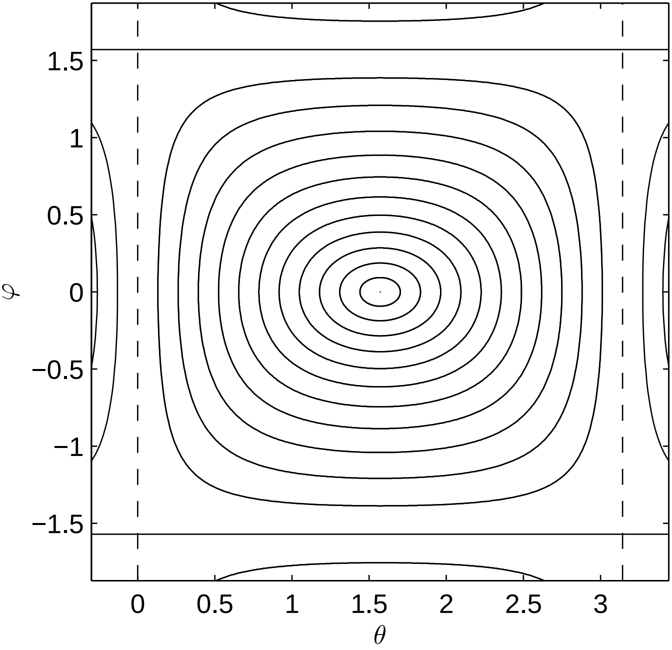
It can be shown that the equations of motion can be reduced to

$$\dot{\theta} = -b \sin w$$
$$\dot{w} = -b \cos\theta \cos w / \sin\theta,$$

where $w = \varphi - \psi$, which, in turn, leads to the separable equation

$$d\theta/dw = \tan\theta \tan w.$$

All the solutions $\big(\theta(t), \varphi(t), \psi(t)\big)$ represent periodic motions.

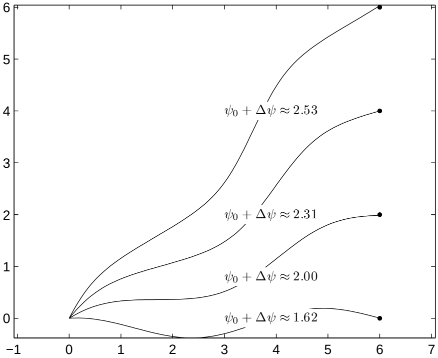


1.5
1
0.5
0
−0.5
−1
−1.5
φ
0
0.5
1
1.5
2
2.5
3
θ

The periodicity of $\big(\theta(t), \varphi(t), \psi(t)\big)$ implies that $z(t)$, $u(t)$ and $v_0(t)$ are periodic, and so is $\omega = v_0 z + z \times \dot{z}$. The trajectory $x(t)$ is determined by its initial value $x(t_0)$, the initial conditions $(\theta_0, \varphi_0, \psi_0)$ and the energy level. The no-sliding condition $\omega \times \mathbf{e}_3 = \dot{x}$ implies that $\dot{x}(t)$ is also periodic and therefore $x(t)$ is the *superposition of a periodic motion and a uniform translation*.

Important remark

Preservation of the energy μ *is compatible with an instantaneous change of* ψ. This also means that such an instantaneous change will preserve the manifold N on which the system evolves.



Calculated Impulse for the Plate-Ball System.

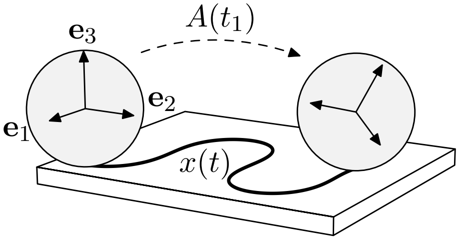
We implement physically an instantaneous change $\Delta\psi$ as an instantaneous elastic impulse.

Compute the increment Δu and Δv_0 due to the change of (φ, θ, ψ) to $(\varphi, \theta, \psi + \Delta\psi)$. Then $\Delta\omega = (\Delta v_0)z - \Delta u$. To calculate the impulse of the instantaneous horizontal impact applied by the upper plate on the top of the ball we use the equation of balance of momentum, to obtain an impulse $(1/2)\big((1 + M)\Delta\omega - (1 - I_3)(\Delta v_0)z\big) \times \mathbf{e}_3$.

Outline

Optimal control

- Take a fixed time interval, and give (x_0, A_0), (x_1, A_1) (initial and final configurations).
- Among all paths $x(t)$ on the ground that make the ball go from orientation A_0 to orientation A_1, find the shortest one (local extremum).



Equations for optimal curves

The equations for the optimal trajectories $x(t)$ on the ground are

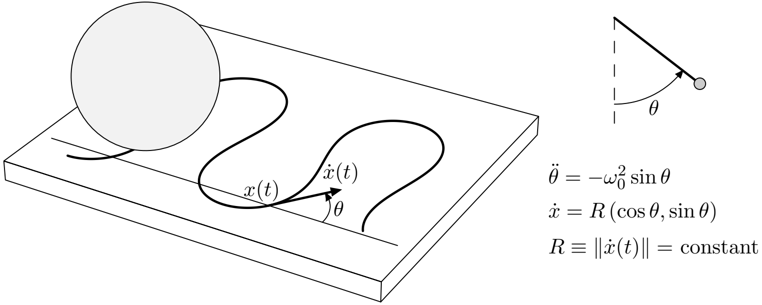
$$\dot{v} = (\mathbf{e}_3 \times \dot{x}) \times v$$
$$\ddot{x} = v^3 \mathbf{e}_3 \times \dot{x}.$$

where $v = v^1\mathbf{e}_1 + v^2\mathbf{e}_2 + v^3\mathbf{e}_3$ is a Lagrange multiplier.

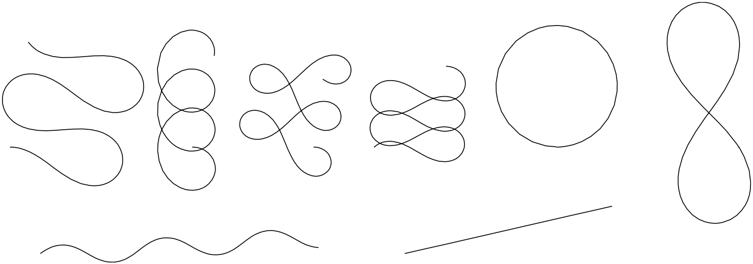
From the equations of motion, it is possible to see that $\|\dot{x}\|$ is a constant. Write $x(t) = R(\cos\theta, \sin\theta)$. Then the equations of motion are equivalent to

$$\ddot{\theta} = -\omega_0^2 \sin\theta$$

which is the equation for a pendulum. Here $\omega_0 = g/l$, where l and g are the length and the acceleration of gravity for the pendulum system, respectively.



Some typical solutions



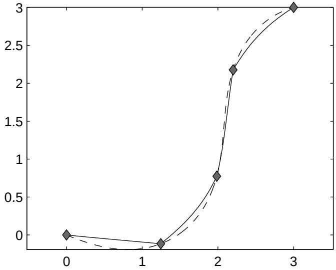
Outline

Impulsive control

Suppose that one of these optimal solutions mentioned before is given.

The goal is to try to make the ball follow the given path, by applying a series of impulses that do not change its kinetic energy (elastic impacts).

Select $N+1$ consecutive points $y_0 = x_0,\ y_1, \ldots, y_N = x_1$ along this trajectory. Starting from initial configuration of the ball we join each pair (y_k, y_{k+1}), $k = 0 \ldots N-1$ in sequence, by free trajectories.



By refining the partition, it is reasonable that one should obtain an arbitrarily close impulsive control approximation to the solution of the isoparallel problem. How to estimate the error?

We will mention some results leading to the following fact:

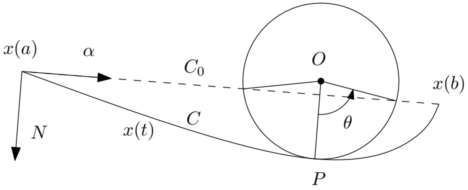
- Consider two paths joining the same points. The difference in the resulting reorientations is bounded by the area enclosed between the paths *on the ground*.

Lemma. Let C be a curve in $\mathbb{R}^2$ given by $x(t)$, $t \in [a, b]$. Let α and N be as in the figure. Write

$$x(t) = x(a) + \varphi(t)\alpha + \psi(t)N.$$

Assume that $\dot{\varphi}(t) > 0$ and $\psi(t) > 0$, and, also, $0 < k(t) < \tilde{k}$ for $t \in (a, b)$, where $k(t)$ is the curvature of C. Consider the uniquely defined circle of radius $\tilde{\rho} = 1/\tilde{k}$ and center $O = O_\alpha \alpha + O_N N$ that contains the only point $P \in C$, $P = x(a) + \varphi(\overline{t})\alpha + \psi(\overline{t})N$, such that $\psi(\overline{t})$ is the only maximum of $\psi(t)$, and, besides, $\langle P - O, \alpha \rangle = 0$. Finally, assume that $O_N < 0$. Then

$$\psi(\overline{t}) \leq \frac{\tilde{k}|x(b) - x(a)|^2}{4}.$$



Lemma. Let C be a curve as in the previous lemma, and let $A(t) \in \mathrm{SO}(3)$, $t \in [a, b]$ be the reorientation that results from rolling the ball along C to time t. Define the family of curves C_λ as given by $x_\lambda(t) = x(a) + \varphi(t)\alpha + \lambda\psi(t)N$, so $C_1 = C$ and C_0 is a straight line. Let $A_\lambda(t)$ be the reorientation corresponding to rolling the ball along C_λ to time t. Then $A_\lambda(b)$ is a curve on $\mathrm{SO}(3)$ with velocity vector $dA_\lambda(b)/d\lambda = \xi_\lambda A_\lambda(b)$, where

$$\xi_\lambda = \int_a^b (A_\lambda(t))^{-1} \left(\frac{\partial x_\lambda(t)}{\partial \lambda} \times \frac{\partial x_\lambda(t)}{\partial t} \right) dt$$

is in the Lie algebra $\mathfrak{so}(3) \equiv \mathbb{R}^3$ of $\mathrm{SO}(3)$.

Lemma. Using the Euclidean norm for matrices, we have $|A(b) - A_0(b)| \leq \tilde{k}|x(b) - x(a)|^3/4$.

Proof.

$$|A(b) - A_0(b)| \leq \left| \int_0^1 \xi_\lambda \, d\lambda \right| \leq \int_0^1 \int_a^b \left| (A_\lambda(t))^{-1} \left(\frac{\partial x_\lambda(t)}{\partial \lambda} \times \frac{\partial x_\lambda(t)}{\partial t} \right) \right|$$

$$\leq \int_0^1 \int_a^b \left| \frac{\partial x_\lambda(t)}{\partial \lambda} \times \frac{\partial x_\lambda(t)}{\partial t} \right| \, dt \, d\lambda,$$

which is the area of the region between C and the straight line C_0. This region is inside a rectangle of length $|x(b) - x(a)|$ and height $\psi(\bar{t}) \leq \frac{\tilde{k}|x(b)-x(a)|^2}{4}$, so the lemma follows.

Long trajectories

Thus, given a trajectory $x(t)$ of length l we partition it in n portions. Assume that each portion has length l/n. Let $A(t_1)$ be the final orientation corresponding to $x(t)$ and let $A_0(t_1)$ be the orientation obtained with *straight lines* joining the endpoints of the n portions in sequence. If $\tilde{k}$ is the maximum (positive) curvature of $x(t)$, then

$$|A(t_1) - A_0(t_1)| \leq \sum_{i=1}^{n} \frac{\tilde{k}}{4} |x(s_{i-1}) - x(s_i)|^3 \leq \frac{\tilde{k} l^3}{4n^2}.$$

Now, if instead of joining the endpoints of these portions using straight lines we use the trajectories of the symmetric ball rolling **according to its dynamics**, we can perform a similar analysis to estimate the difference between $A_0(t_1)$ and the resulting final reorientation $A_{\text{dyn}}(t_1)$. Then $|A_{\text{dyn}}(t_1) - A(t_1)|$ will be bounded by a constant times l^3/n^2.

Some final comments

One of our goals is staying close to the optimal curve, precisely because its length is a minimum (or extremum). Under these conditions, it is not possible, in general, to obtain the *exact* final reorientation.

The reorientation map from the (finite dimensional) space of impulsive trajectories into $\mathrm{SO}(3)$ has poor regularity conditions. In particular, it does not have full rank for straight lines. This can be checked using the curvature of the connection. In addition, its image (for small deformations) looks like a half-space.

Even when it has full rank, images of nice open sets in the space of trajectories are very thin open sets of $\mathrm{SO}(3)$ and the desired reorientation can fall outside them. These sets get even thinner as we refine the partition and take portions of the path that look increasingly similar to straight lines.

Further reading

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